Fitting Hyperboloid and Hyperboloid Structures

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Abstract:
In this paper, we present the design of hyperboloid structures and techniques for hyperboloid fitting which are based on minimizing the sum of the squares of the algebraic distances between the noisy data and the hyperboloid. Algebraic fitting methods solve the linear least squares (LS) problem, and are relatively straightforward and fast. Fitting orthogonal hyperboloid is a difficult issue. Usually, it is impossible to reach a solution with classic LS algorithms, because they are often faced with the problem of convergence. Therefore, it is necessary to use special algorithms e.g. nonlinear least square algorithms. Hyperboloid has a complex geometry as well as hyperboloid structures have always been interested. The two main reasons, apart from aesthetic considerations, are strength and efficiency.

Keywords:
Hyperboloid Structure, Design of Hyperboloid, Fitting Hyperboloid, Least Squares Problem

1. Introduction

The industrial manufacturing is making more widespread use of three-dimensional object recognition techniques. The shapes of many 3D objects can be described using some basic surface elements. Many of these basic surface elements are quadric surfaces, such as spheres and cylinders. For some industrial parts, the surface located at the junction of two other surface elements can often be regarded as either a hyperbolic surface (concave) or a parabolic surface (convex). Therefore, the recognition of hyperboloids and paraboloids is necessary for the recognition or inspection of such kinds of 3D objects. The objective of surface fitting is to fit some kind of mathematical model, such as an equation under a certain kind of coordinate system, to the sample data. For centered quadric surfaces, such an equation can be expressed in a form where fit parameters have a clear geometric meaning, such as a distance or a rotation angle Min and Newman (1970).

As hyperboloid structures are double curved, that is simultaneously curved in opposite directions, they are very resistant to buckling. This means that you can get away with far less material than you would otherwise need, making them very economical. Single curved surfaces, for example cylinders, have strengths but also weaknesses.

Double curved surfaces, like the hyperboloids in question, are curved in two directions and thus avoid these weak directions. This means that you can get away with far less material to carry a load, which makes them very economical. The second reason, and this is the magical part, is that despite the surface being curved in two directions, it is made entirely of straight lines. Apart from the cost savings of avoiding curved beams or shuttering, they are far more resistant to buckling because the individual elements are straight URL-1 [17], URL-2 [18].

This is an interesting paradox: you get the best local buckling resistance because the beams are straight and the best overall buckling resistance because the surface is double curved. Hyperboloid structures cunningly combine the contradictory requirements into one form.

The hyperboloid is the design standard for all nuclear cooling towers and some coal-fired power plants. It is structurally sound and can be built with straight steel beams.

Figure 1(a). *Shukhov Tower Nizhny Novgorod 1896.*

Figure 1(b). *Canton Tower, Guangzhou, China © Kenny Ip.*

When designing these cooling towers, engineers are faced with two problems:

(1) the structure must be able to withstand high winds and

(2) they should be built with as little material as possible.

The hyperbolic form solves both of these problems. For a given diameter and height of a tower and a given strength, this shape requires less material than any other form.

The pioneer of hyperboloidal structures is the remarkable Russian engineer V. Shukhov (1853-1939) who, among other accomplishments, built a hyperboloidal water tower for the 1896 industrial exhibition in Nizhny Novgorod. Hyperboloidal towers can be built from reinforced concrete or as a steel lattice, and is the most economical such structure for a given diameter and height. The roof of the McDonnell Plantarium in St. Louis, the Brasilia Cathedral and the Kobe Port tower are a few recent examples of hyperboloidal structures. The most familiar use, however, is in cooling towers used to cool the water used for the condensers of a steam power plant, whether fuel burning or nuclear. The bottom of the tower is open, while the hot water to be cooled is sprayed on wooden baffles inside the tower. Potentially, the water can be cooled to the wet bulb temperature of the admitted air. Natural convection is established due to the heat added to the tower by the hot water. If the air is already of moderate humidity when admitted, a vapor plume is usually emitted from the top of the tower. The ignorant often call this plume "smoke" but it is nothing but water. Smokestacks are the high, thin columns emitting at most a slight haze. The hyperboloidal cooling towers have nothing to do with combustion or nuclear materials. Two such towers can be seen at the Springfield Nuclear Plant on The Simpsons. The large coal facility at Didcot, UK also has hyperboloidal cooling towers easily visible to the north of the railway west of the station. Hyperboloidal towers of lattice construction have the great advantage that the steel columns are straight URL-1 [17]. For hyperboloid application please refers to URL-3 [19], URL-4 [20].

The paper has six parts. We will first give some general information about hyperboloid structures, the basic definition of Hyperboloid starts with giving mathematical equations to explain the concepts. Then it reviews the extend literature relevant to hyperboloid fitting. To show how to best fitting Hyperboloid, are carried out, we solve this problem separately: algebraic direct fitting Hyperboloid and the best fitting Hyperboloid. The efficacy of the new algorithms is demonstrated through simulations. The paper concludes with a discussion of theoretical and managerial implications and directions for further research.

Unfortunately, best fitting hyperboloid has not been discussed in literature. However, most of the few fitting techniques in the literature are algebraic fitting are not orthogonal fitting [1].

There are some published methods that focus on this topic. Hall approach is a least squares based approach to recover the parameters of quadric surfaces from depth maps [7]. Although parameter recovery for quadric surfaces might seem to require a nonlinear method, the quadric coefficients can be recovered by least squares and then the geometric parameters of the surface can be recovered using Hall's approach. Cao et al.'s method is another pure numerical approach which emphasizes minimization of overall errors represented by a certain kind of approximate orthogonal distance [18]. Their approach uses the iterative Newton method to perform nonlinear optimization. Their method may terminate at local minima.

We could not find enough studies with numerical examples in the literature. Some of author hides data and/or results. Since no other comparable orthogonal fitting hyperboloid application could be found in literature. Against this background, the purpose of the study is to give an orthogonal fitting hyperboloid with a numerical example. In this article, we demonstrate that the geometric fitting approaches, provides a more robust alternative then algebraic fitting-although it is computationally more intensive.

The most time consuming part of fitting hyperboloid is the calculation of the shortest (orthogonal) distances between each point and the hyperboloid. When we look at literature in this regard, we often see studies about ellipsoidal distances. We can develop a distance finding algorithm for hyperboloid by simulating it. And we did so, but this process is a little more difficult than the ellipsoid. In hyperboloids, one or two semi-axis is negatives, which changes the order of magnitude of the hyperboloid semi-axis. In the literature on shortest distances from an ellipsoid we see the various studies**:** [8], [9], [10], [12], [3]**.** For the solution [8] gives a method that is based on sixth degree polynominal. He has benefited from the largest root of 6th degree polynomial**.** [10] Gives a vector-based iteration process for finding the point on the ellipsoid. [12] Claims his method turns out to be more accurate, faster and applicable than Feltens method. [3] is a little more advanced than the [12]. A recent work on the subject gives us useful information on this topic (the orthogonal distance to Hyperboloid) for detailed information on this subject refer to[5].

2. Materials and Methods

2.1. Design of Hyperboloid

Hyperboloid a geometric surface consisting of one sheet, or of two sheets separated by a finit distance who sections parallel to the three coordinate planes are hyperbolas or ellipses. The hyperboloid of one sheet is possibly the most complicated of all the quadric surfaces. For one thing, its equation is very similar to that of a hyperboloid of two sheets, which is confusing. For another, its cross sections are quite complex. The standard equation of an hyperboloid centered at the origin of a cartesian coordinate system and aligned with the axis is given below.

Let a hyperboloid be given with the three semi axis a, b, c (i.e., the lengths of the real long axis, real short axis and the imaginary axis respectively) see Fig.1

Hyperboloid equation

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} - \frac{Z^2}{c^2} = \pm 1 \qquad (1)$$

+1 where on the right hand side of (1) corresponds to a hyperboloid of one sheet, on the right hand side of -1 to a hyperboloid of two sheets.

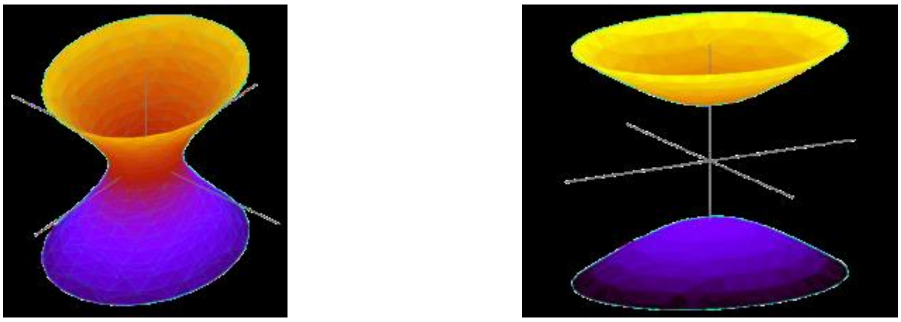

Figure 2. *Hyperboloid*

a) Hyperboloid of one sheet)Hyperboloid of two sheets

2.2. Generalised Equation of Hyperboloid

More generally, an arbitrarily oriented hyperboloid, centered at **v**, is defined by the equation

$$(x - v)^T A (x-v) = 1 \qquad (2)$$

where A is a matrix and **x**, **v** are vectors. The eigenvectors of A define the principal directions of the hyperboloid and the eigenvalues of A are the reciprocals of the squares of the semi-axis: $1/a^2$, $1/b^2$ and $1/c^2$. The one-sheet hyperboloid has two positive eigenvalues and one negative eigenvalue. The two-sheet hyperboloid has one positive eigenvalue and two negative eigenvalues [11].

3. Fitting Hyperboloid to Noisy Data

Although hyperboloid equation is quite simple and smooth but computations are quite difficult on the hyperboloid. Generally a hyperboloid is defined 9 parameters. These parameters are; three coordinates of center (X_o, Y_o, Z_o), three semi-axis (a_x, a_y, b) and three rotational angles (ε,ψ,ω) which represent rotations around x-,y- and z- axis respectively (Fig.2).These angles control the orientation of the hyperboloid. For this minimization problem to have a unique solution the necessary conditions is to be $n>=$ 9 and the data points lie in general position (e.g., not all data points should lay is an elliptic plane). Throughout this paper, we assume that these conditions are satisfied. For the solution of the fitting problem, the linear or linearized relationship, written between the given data points and unknown parameters (one equation per data points), consists of equations, including unknown parameters [4].

3.1. Algebraic Direct Hyperboloid Fitting Methods

Algebraic direct fitting methods are a standard class of methods commonly used for fitting quadric surfaces. Algebraic fitting uses a generalized eigenvalue method.

The general equation of a hyperboloid is given as

$$A x^2 + B y^2 + C z^2 + 2 D xy + 2 E xz + 2 F yz + 2 G x + 2 H y + 2 I z - 1 \quad (3)$$

Algebraic methods are a linear LS problem. It is solving directly easily. Given the data set $((x, y, z)_i$, $i=1,2,\ldots,n)$, the fitted hyperboloid by obtaining the solution in the LS sense of the linear algebraic equations.

ith row of the nx9 matrix

$$[\ x_i^2 \quad y_{i.}^2 \quad z_i^2 \quad 2x_i\, y_i \quad 2x_iz_i \quad 2y_iz_i \quad 2x_i \quad 2y_i \quad 2z_i\ ] \quad (4)$$

it is solved easily in the LS sense as below

Or it is solved easily as MATLAB expression:

$$p = [\ x.^2 \quad y..^2 \quad z.^2 \quad 2x.\, y. \quad 2x.z. \quad 2y.z. \quad 2x \quad 2y \quad 2z\ ] \backslash \ ones(n) \quad (5)$$

$ones\ (n)$ =[11 … 1]^T

This fitting algorithm, we need to check whether a fitted shape is a hyperboloid. In theory, the conditions that ensure a quadratic surface to be a hyperboloid have been well investigated and explicitly stated in analytic geometry textbooks. Since a hyperboloid can be degenerated into other kinds of elliptic quadrics, such as an elliptic paraboloid. Therefore a proper constraint must be added [4].

$$m = \begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix} \qquad M = \begin{bmatrix} A & D & E & G \\ D & B & F & H \\ E & F & C & I \\ G & H & I & -1 \end{bmatrix}$$

$$\rho_3 = rank\ m \qquad \rho_4 = rank\ M \qquad \Delta = det\ M \tag{6}$$

And k_1, k_2 and k_3 are the roots of

$$\begin{vmatrix} A-x & D & E \\ D & B-x & F \\ E & F & C-x \end{vmatrix} = 0 \tag{7}$$

The following conditions must be met for hyperboloid [2].

$\rho_3 = 3$, $\rho_4 = 4$, $sign\ (\Delta) = +$

and at least one of the signs of roots (k) must be different.

In this paper, we assume that these conditions are satisfied. Algebraic methods all have indisputable advantage of solving linear LS problem. The methods for his are well known and fast. However, it is intuitively unclear what it is we are minimizing geometrically Eq. (3) is often referred to as the "algebraic distance" to be minimized [13].

4. Obtaining Hyperboloid Parameters from Conic Equation

After conical equation Eq. (3) was determined. This section we determines the center, semi-axis and rotation angles of the hyperboloid.

$$\begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix} \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} + \begin{bmatrix} G \\ H \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

The solution of above equation system which is established conic coefficients gives us the coordinates of hyperboloid's center (X_o, Y_o, Z_o).

For finding of semi-axis (a,b,c) and rotation angles of the hyperboloid (ε, ψ, ω):

Firstly eigenvalues and eigenvectors of above coefficient matrix (S_{3x3}) in Eq. (8) can be easily calculated with the following MATLAB command

[evecs, evals]= eig (S)

$$evals\text{: Eigenvalues of } (S) = [\lambda_1\ \ \lambda_2\ \ \lambda_3]^T \tag{9}$$

Let $\lambda 1$, $\lambda 2$ and $\lambda 3$ get the eigenvalues of the matrix S, in descending order Semi-axis of hyperboloid (a,b,c) obtained the eigenvalues of S as belov

$$a = sign(\lambda_1)/\sqrt{abs(\lambda_1)} \quad b = sign(\lambda_2)/\sqrt{abs(\lambda_2)} \quad c = sign(\lambda_3)/\sqrt{abs(\lambda_3)} \tag{10}$$

It should not be forgotten that the one-sheet hyperboloid has two positive eigenvalues and one negative eigenvalue. The two-sheet hyperboloid has one positive eigenvalue and two negative eigenvalues. So it is necessary to pay attention to this in the rooting process. Rooting can be done as above.

evecs: Eigenvectors of (S) give us R - rotatiton angles of hyperboloid.

The following link can be used for hyperboloid parameter from the conic equation URL-6 [22].

5. Numerical Example and Discussion

For numerical applications 20 triplets (x, y, z) cartesian coordinates were produced.

Here data points coordinates and results as shown Table-1

Find the best fitting hyperboloid for the given noisy data is based on both of algebraic and orthogonal methods

x:	-4	-4	-4	-4	-4	-4	-4	-2	-2	-2	-2	1	1	4	4	4	4	4	4	4
y:	-8	-6	-3	0	2	5	8	-8	-6	5	8	-8	8	-8	-6	-3	0	2	5	8
z:	6	6	4	3	3	4	6	1	-4	-9	-2	-2	-7	3	-1	-4	-7	-6	-3	1

We show both the algebraic and orthogonal fitting results are as shown in Table 1.

Table 1. *The result of direct fitting hyperboloid.*

Algebraic direct fitting							
Center of Coordinates			Semi-axis			Rotational matrix	RSS*
x_0	y_0	z_0	**a**	**b**	**c**		
0.657	0.445	-8.926	3.2408	8.0121	-10.568	0.0078 0.0074 -1 0.0115 1 0.0075 1 -0.0116 0.0077	1.4721

Parameter of Algebraic direct fitting:

0.05745 0.0094013 -0.0053975 -0.0003616 -0.00048776 -0.000168 -0.041941 -0.0054448 -0.047784

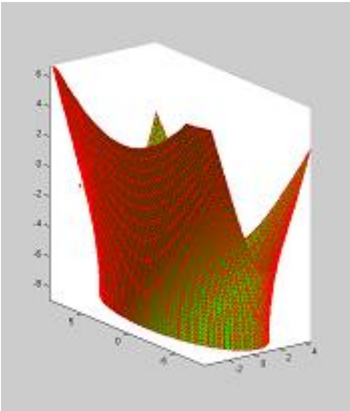

Figure 3. *Fitting result.*

6. Discussion and Suggestions

It is also seen in our numerical application that the RSS indicator, which is a quality indicator is 1, 4721. Our experience tells us that if the coordinates of given points consists of a large number this will cause bad condition. Therefore, before fitting, you must shift the given coordinates to the center of gravity, after fitting operation the coordinates of hyperboloid's center must be shifted back to the previous position.

In this study, there is no mention of the accuracy (variances) of the parameters obtained from the fitting and, consequently, of the accuracies of the estimated geometric elements of the hyperboloid, because the geometric elements of the hyperboloid are obtained indirectly. We have achieved first conical parameters after that the geometric parametric of the hyperboloid. Hence the accuracy calculations require long study. Therefore, we consider the accuracy calculations can be subject a separate subject study.

7. Conclusions

In this paper we study on the algebraic fitting hyperboloid. From the results, it is apparent that the orthogonal fitting hyperboloid always exhibits less RSS error than the algebraic direct fitting hyperboloid. The problem of fitting hyperboloid is encountered frequently in the image processing, in the modeling of some industrial parts, computer games, etc. The paper has presented a new method of orthogonal fitting hyperboloid. It has been compared to the other existing methods. In conclusion, the presented method may be considered as fast, accurate and reliable and may be successfully used in other areas. The presented fitting algorithm can be applied easily for ellipsoid, and sphere also other surface such as paraboloid.

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