Volume 4, Issue 3, 2020 ISSN: 2617-4553
DOI: https://doi.org/10.31058/j.er.2020.43001

Exact Analytical Solutions in Closed Recurrent form for the Non-Stationary Linear Inverse Heat Conduction Problem for Bodies of One-Dimensional Geometry with Boundary Conditions on One And Two Surfaces

Lobanov Igor Evgenievich[1*]

[1] Technical Sciences, Moscow Aviation Institute, Moscow, Russia

Email Address

grigorchuk.grigorchuck@yandex.ru (Lobanov Igor Evgenievich)
*Correspondence: grigorchuk.grigorchuck@yandex.ru

Received: 25 December 2019**; Accepted:** 17 February 2020**; Published:** 22 July 2020

Abstract:

In this paper, we obtained exact analytical solutions for the non-stationary linear inverse heat conduction problem for bodies of one-dimensional geometry with boundary conditions on one surface, as well as on two surfaces for a plane body, a hollow cylinder, and a hollow sphere, obtained in a closed recurrent form. The recurrent form of the solution of the non-stationary linear inverse heat conduction problem for bodies of one-dimensional geometry with boundary conditions on one surface, as well as on two surfaces for a plane body, hollow cylinders and spheres, presented in the article is a solution in a closed form from a single position, which is not always perhaps explicitly.

Keywords:

Thermal Conductivity, Analytical, Non-stationary, Linear, One-dimensional, Inverse Problem, Surface, Border Conditions, Unilateral, Bilateral, Recurrent , Flat, Spherical, Cylindrical

1. Introduction

N Internet Direct mathematical modeling allows prediction the thermal esostojanie state wide range of operation modes, for example, a technical system, to analyze the influence of various factors on the behavior of the system and select the optimum thermal conditions.

The use of direct methods of mathematical modeling requires an analysis of the accuracy of mathematical models. The model can have a very complex structure and take into account a fairly large number of factors. However, it is necessary to set the numerical values of all members of the model characteristics, in particular, thermal properties of materials, characteristics of the thermal interaction with the flushing medium and others. If the information is missing or has low accuracy, then the complex mathematical model loses its advantages and does not provide the required accuracy of the forecast of thermal conditions.

The practical application of mathematical modeling of heat transfer shows that the possible unsatisfactory accuracy in mathematical modeling, for example, of high-intensity thermal processes is due to the low accuracy of determining characteristics using traditional direct methods [19]. In such cases, it can be very effective to use computational and experimental methods that are based on the principles of identification of systems with distributed parameters, which are based on algorithms and methods for solving various types of incorrect inverse heat transfer problems [19].

As is known, in direct problems, the desired field is the temperature field, which is found as a solution of the heat equation with known parameters of internal transfer, corresponding to the known boundary and initial conditions, and in the inverse problems of thermal conductivity, the initial temperature distribution and boundary conditions are unknown functions to be determined.

Inverse problems are divided into two main types:

a. Characterization of internal energy transfer - warm coefficients o- and thermal diffusivity , specific heat, light absorption coefficients and the like are conductive Xia physical characteristics of the substance;

b. determination of the conditions of the external energy exchange between the body and the medium, i.e., finding the boundary conditions: calculation of the temperature of the outer surface and the heat flux passing through it, calculation of variable heat transfer coefficients, thermal contact resistances, degrees of blackness, angular coefficients of radiation, from the south include the position of the surface of the phase transition or destruction, the preparation of unsteady power and energy balances, etc. [19].

It is clear that it is much more difficult to obtain a solution to the inverse heat conduction problem than the direct one; however, in the direct problem, when measuring or realizing the given boundary conditions, many experimental obstacles may arise. Physical conditions are, for example, such that it is practically not always possible to install the sensor on the surface of the body or the measurement accuracy is significantly reduced due to the placement of sensors. Therefore, it is often difficult to measure the law of the temperature of a heated solid surface. It is much easier to perform a sufficiently accurate time measurements of temperature dependency of internal points in a thermally insulated body surface. Thus, the problem arises of choosing between relatively inaccurate measurements and a complex analytical problem. At the same time, a sufficiently accurate and easily implemented solution of the inverse problem would simultaneously reduce both difficulties to a minimum [19]. The direct heat conduction problem under correctly posed conditions has a unique solution. In the case of inverse problems, the identity of temperature fields is possible as a result of external influences that are different in nature, but equivalent in energy terms [5,6,19].

The temperature field of a solid does not uniquely determine the boundary conditions under which it arose. A number of energy equivalent to their exposure meters on the system of boundary conditions may vary to reflect the complex thermal processes.

An example is the fact that any redistribution of heat flux densities, for example, between convective and radiation components when combined, leads to an identical thermal state of the system [19].

There are other drawbacks inherent to the inverse methods of studying unsteady heat transfer in technical systems: limitation of the number of points in parts in which temperatures and heat fluxes are measured; certain experimental values of temperatures and heat flows, on which the calculations are made, contain measurement errors even when using precise devices, as occupancy sensors in a solid body in some way violates the temperature field detail her; curvature of the surface, the spatial and temporal variation of heat flow in the body does not make it possible to accurately predict the direction of heat flow, or other layers you define the place Raspaud decomposition sensor, which should be normal to the surface.

It should be noted that reverse methods do not allow the physical interpretation of non-stationary complex processes occurring in systems.

In addition to the disadvantages, including the above, inverse methods have some advantages in comparison with the straight. In a direct problem, when measuring or realizing given boundary conditions, many experimental obstacles may arise. The physical conditions in the test systems may be such that the setting cannot be on the surface of the sensor body (e.g., coatings on the surface) or substantially reduced measurement accuracy due to the placement of sensors, therefore, it is often difficult to measure the law of changes in temperatures and heat flows of solid surfaces.

Summarizing the above, we can conclude that there is a relevant preparation in the form of single exact analytical solutions closed nonstationary linear inverse heat conduction problem for body-dimensional geometry with one boundary conditions on the second surface. In this article the exact closed analytical solution of the inverse heat conduction problem is achieved in a recurrent form, that is, in an implicit form, since this is not always possible in an explicit form [2,3,4,5,6].

2. Recurrent Solutions for Non-Stationary Linear Inverse Problems and Heat Conduction for Bodies of One-Dimensional Geometry with Boundary Conditions on one Surface

There is an exact solution of inverse problems of unsteady heat conduction relatively few in number, and their significantly smaller than the corresponding decisions of direct non-stationary heat conduction problems. You can specify that one-quarter and of the first successful attempts to solve the inverse transient heat conduction problems for a flat body was first made in 1890 Y.Stefan [1].

Subsequently, solutions were obtained for the one-dimensional linear inverse non-stationary heat conduction problem in an independent manner O.R.Burggraf [2] and D.Lengford [3] and the assumption of become widely known at the sensor locations transient heat flux density and tempera tours. Exact solutions for temperature fields from previously known temperatures at two different internal points using the Laplace transform method were obtained by M. Imber and D. Khan [4].

A nalogichnye solutions for one-dimensional bodies prefedenes as in papers [5,6], in which solutions for a non-temperatures are given explicitly, a thermal density of current deterministic differentiation floor s tempera round.

Subsequently, solutions to similar problems were obtained, partly having not only theoretical but also applied character, including the nonlinear one-dimensional problem of unsteady heat conduction [7-19].

As was partially indicated in [2-6], the expression of solutions for the non-stationary linear inverse heat conduction problem for bodies of one-dimensional geometry in explicit form is not possible in all cases, therefore, in order to obtain the final solution, additional assumptions have to be applied, for example, as in [2] where the thin wall assumption is used.

The purpose of this article is to obtain a solution to the non-stationary linear inverse heat conduction problem for bodies of one-dimensional geometry with boundary conditions on the same surface from common positions in a closed recursive form, which will have distinct advantages over solutions, since they can be obtained for all the above problems, but not explicitly for everyone.

We write the equation of nonlinear non-stationary thermal conductivity for one-dimensional geometry and constant curvature (in this case, the radial coordinate is considered) in the following form [5]:

$$\frac{1}{a}\frac{\partial t}{\partial \tau} = \nabla^2 t = r^{1-k}\frac{\partial}{\partial r}\left(r^{k-1}\frac{\partial t}{\partial r}\right) = \frac{\partial^2 t}{\partial r^2} + \frac{k-1}{r}\frac{\partial t}{\partial r} \tag{1}$$

where k is the number of final measurements: 1 is a flat field; 2 - cylindrical; 3 - spherical; t is the temperature; r is the radial coordinate; *and* - coefficient of thermal diffusivity.

The definition domain of differential equation (1) is from 0 to r_2 (in this case, the radial coordinate is considered) in coordinate (in the case of hollow bodies: from r_1 (radial coordinate of the inner surface) to r_2) and from 0 to the current value τ in time ($\tau > 0$).

In a dimensionless form, this equation can be written as follows [5]:

$$\frac{\partial T}{\partial Fo} = \frac{\partial^2 T}{\partial \rho^2} + \frac{k-1}{\rho}\frac{\partial T}{\partial \rho} \tag{2}$$

where $\mathrm{Fo} = \frac{a\tau}{r_1^2}$ is the Fourier criterion; T is the dimensionless temperature; $\rho = r / r_1$ is the dimensionless coordinate; r_1 - radial coordinate at which the boundary conditions are specified.

Inverse heat conduction problem for equation (1) or (2) is to find the boundary conditions on the one-dimensional body surface at known unsteady temperature and heat flow and thermal characteristics of the body material is not temperature dependent.

In the framework of this article, we study the process of heat conduction at a time quite remote from the initial moment of time; therefore, the influence of the initial conditions practically does not affect the temperature distribution at the time of measurement or observation (the so-called “task without initial conditions”). In practical terms, this may mean that, with a sufficient distance from the initial moment of time, the aftereffect component, taking into account the influence of the initial conditions, becomes so small that it will already be less than the measurement error of sensors measuring temperatures and heat fluxes [5,6].

The component of the influence of the temperature field of a one-dimensional layer, which is heated on the inner surface, is considered using a dimensionless coordinate, for which the heated surface corresponds to a unit value (the homochronism complex refers to this internal radial coordinate) can be represented as follows [5]:

$$T(\rho, Fo) = \sum_{n=0}^{\infty} T_1^{(n)}(Fo)\, P_{n,1} + \sum_{n=0}^{\infty} Ki^{(n)}(Fo)\, P_{n,2} = \sum_{n=0}^{\infty} \Theta_{n,1}\, P_{n,1} + \sum_{n=0}^{\infty} \Theta_{n,2}\, P_{n,2} \quad (3)$$

where is $\mathrm{Ki} = \frac{qr_1}{\lambda\Delta t}$ the Kirpichev criterion; $\mathrm{Fo} = \frac{a\tau}{r_1^2}$- Fourier criterion; $\rho = r / r_1$ is the dimensionless coordinate; r_1 -radial coordinate on which boundary conditions are specified; a is the coefficient of thermal diffusivity; λ is the coefficient of thermal conductivity; q is the heat flux density; Δt is the temperature difference.

On the heated surface there is a boundary condition of the second kind. In this case, the heat flux density and temperature are measured on the same surface.

Solutions for bodies of simple configuration will differ in the values of the radial quasi-polynomials $P_{n,1}$ and $P_{n,2}$.

In the framework of this work, these quasipolynomials will be solved in recurrent forms, in contrast to [2,3,4,5,6] and [7-19].

2.1. Flat Plate

The quasi-polynomials $P_{n,1}$ and $P_{n,2}$ for a flat plate will be as follows:

$$P_{n+1,1} = \int_0^{\rho} \int_0^{\rho} P_{n,1}\, d\rho d\rho \quad (4)$$

$$P_{n+1,2} = \int_0^{\rho} \int_0^{\rho} P_{n,2}\, d\rho d\rho \quad (5)$$

$$P_{0,1} = 1; P_{0,2} = \rho \quad (6)$$

For the first quasi-polynomials $P_{1,1}$ and $P_{2,1}$, etc. , $P_{1,2}$ and $P_{2,2}$, etc. for a flat plate, you can write:

$$P_{1,1} = \frac{\rho^2}{2}; P_{2,1} = \frac{\rho^4}{24}; P_{3,1} = \frac{\rho^6}{720}; \ldots; \ldots; \quad (7)$$

$$P_{1,2} = \frac{\rho^3}{6}; P_{2,2} = \frac{\rho^5}{120}; P_{3,2} = \frac{\rho^7}{5040}; \ldots; \ldots; . \quad (8)$$

Therefore, using the method of mathematical induction, it is possible to write quasipolinomes for solving the inverse non-stationary heat conduction problem when setting boundary conditions on the same surface for a flat plate in a recurrent form:

$$P_{n,1} = \frac{\rho^2}{2n\cdot(2n-1)} P_{n-1,1}; \quad (9)$$

$$P_{n,2} = \frac{\rho^2}{2n\cdot(2n+1)} P_{n-1,2}. \quad (10)$$

2.2. Continuous Cylinder

The quasi-polynomials $P_{n,1}$ for a continuous cylinder will be as follows:

$$P_{n+1,1} = \int_0^{\rho} \frac{1}{\rho} \int_0^{\rho} \rho P_{n,1}\, d\rho d\rho \quad (11)$$

$$P_{0,1} = 1. \quad (12)$$

For the first quasi-polynomials $P_{1,1}$ and $P_{2.1}$, etc. for a solid cylinder, you can write:

$$P_{1,1} = \frac{\rho^2}{4}; P_{2,1} = \frac{\rho^4}{64}; P_{3,1} = \frac{\rho^4}{2304}; \ldots; \ldots; . \quad (13)$$

Volume 4, Issue 3, 2020 ISSN: 2617-4553
DOI: https://doi.org/10.31058/j.er.2020.43001

Hence, using the method of mathematical induction, we can write quasi-polynomials for solving inverse transient heat conduction problem by setting the boundary condition on the axis of a solid cylinder in a recurrent form:

$$P_{n,1} = \frac{\rho^2}{4n^2} P_{n-1,1} \tag{14}$$

2.3. Hollow Cylinder

The quasi-polynomials $P_{n,1}$ and $P_{n,2}$ for a hollow cylinder will be as follows:

$$P_{n+1,1} = \int_1^\rho \frac{1}{\rho} \int_1^\rho \rho P_{n,1}\, d\rho d\rho \tag{15}$$

$$P_{n+1,2} = \int_1^\rho \frac{1}{\rho} \int_1^\rho \rho P_{n,2}\, d\rho d\rho \tag{16}$$

$$P_{0,1} = 1;\ P_{0,2} = \ln\rho \tag{17}$$

For the first quasi-polynomials P_1, and $P_{2.1}$, $P_{1.2}$ and $P_{2.2}$, etc. for a hollow cylinder, you can write:

$$P_{1,1} = \frac{1}{4}\rho^2 - \frac{1}{2}\ln\rho - \frac{1}{4} = \frac{1}{4}\rho^2 - \frac{1}{2}P_{0,2} - \frac{1}{4}P_{0,1} \tag{18}$$

$$P_{1,2} = \frac{1}{4}\rho^2 \ln\rho - \frac{1}{4}\rho^2 + \frac{1}{4}\ln\rho + \frac{1}{4} = \frac{1}{4}\rho^2(\ln\rho - 1) + \frac{1}{4}P_{0,2} + \frac{1}{4}P_{0,1} \tag{19}$$

$$P_{2,1} = \frac{1}{64}\rho^4 - \frac{1}{8}\rho^2 \ln\rho + \frac{1}{16}\rho^2 - \frac{1}{16}\ln\rho - \frac{5}{64} = \frac{1}{64}\rho^4 - \frac{1}{2}P_{1,2} - \frac{1}{4}P_{1,1} - \frac{1}{16}P_{0,2} - \frac{1}{64}P_{0,1} \tag{20}$$

$$P_{2,2} = \frac{1}{64}\rho^4 \ln\rho - \frac{3}{128}\rho^4 + \frac{1}{16}\rho^2 \ln\rho + \frac{1}{64}\ln\rho + \frac{3}{128} = \frac{1}{64}\rho^4\left(\ln\rho - \frac{3}{2}\right) + \frac{1}{4}P_{1,1} + \frac{3}{128}P_{0,1} + \frac{1}{4}P_{1,2} + \frac{5}{64}P_{0,2} \tag{21}$$

$$P_{3,1} = \frac{1}{2304}\rho^6 - \frac{1}{128}\rho^4 \ln\rho + \frac{1}{128}\rho^4 - \frac{1}{64}\rho^2 \ln\rho - \frac{1}{256}\rho^2 - \frac{1}{384}\ln\rho - \frac{5}{1152} = \frac{1}{2304}\rho^6 - \frac{1}{4}P_{2,1} - \frac{1}{64}P_{1,1} - \frac{1}{2304}P_{0,1} - \frac{1}{2}P_{2,2} - \frac{1}{16}P_{1,2} - \frac{1}{384}P_{0,2}\ ;\ \ldots;\ \ldots; \tag{22}$$

$$P_{3,2} = \frac{1}{2304}\rho^6 \ln\rho - \frac{11}{13824}\rho^6 + \frac{1}{256}\rho^4 \ln\rho - \frac{1}{512}\rho^4 + \frac{1}{256}\rho^2 \ln\rho + \frac{1}{512}\rho^2 + \frac{1}{2304}\ln\rho + \frac{11}{13824} = \frac{1}{2304}\rho^6\left(\ln\rho - \frac{11}{6}\right) + \frac{1}{4}P_{2,1} + \frac{3}{128}P_{1,1} + \frac{11}{13824}P_{0,1} + \frac{1}{4}P_{2,2} + \frac{5}{64}P_{1,2} + \frac{5}{1152}P_{0,2};\ldots; \tag{23}$$

Therefore, using the method of mathematical induction, it is possible to write quasi-polynomials for solving the inverse non-stationary problem of heat conduction when setting the boundary condition on the inner surface of the hollow cylinder in a recurrent form:

$$P_{n,1} = \frac{1}{((2n)!!)^2}\rho^2 - \sum_{m=0}^{n-1} \frac{1}{\left((2(n-m))!!\right)^2} P_{m,1} - \sum_{m=0}^{n-1} \frac{2(n-m)}{\left((2(n-m))!!\right)^2} P_{m,2} \tag{24}$$

$$P_{n,2} = \left(\ln\rho - \sum_{m=1}^{n} m^{-1}\right)\rho^2 + \sum_{m=0}^{n-1} \frac{1}{\left((2(n-m))!!\right)^2} \sum_{l=1}^{n-m} l^{-1} P_{m,1} + \sum_{m=0}^{n-1} \frac{1}{\left((2(n-m))!!\right)^2} P_{m,2} + \sum_{m=0}^{n-1} \frac{2(n-m)}{\left((2(n-m))!!\right)^2} \sum_{l=1}^{n-m-1} l^{-1} P_{m,2} \tag{25}$$

2.4. Continuous Ball

The quasi-polynomials $P_{n,1}$ for c flat ball will be as follows:

$$P_{n+1,1} = \int_0^{\rho} \frac{1}{\rho^2} \int_0^{\rho} \rho^2 P_{n,1}\, d\rho d\rho \tag{26}$$

$$P_{0,1} = 1 \tag{27}$$

For the first quasi-polynomials $P_{n,1}$, etc. for a solid ball, we can write:

$$P_{1,1} = \frac{\rho^2}{6}; P_{2,1} = \frac{\rho^4}{120}; P_{3,1} = \frac{\rho^4}{5040}; \ldots; \ldots; \tag{28}$$

Therefore, using the method of mathematical induction, it is possible to write quasi-polynomials for solving the inverse non-stationary heat conduction problem with a boundary condition in the center of a continuous ball in a recursive form:

$$P_{n,1} = \frac{\rho^2}{2n\cdot(2n+1)} P_{n-1,1} \tag{29}$$

2.5. Hollow Ball

The quasi-polynomials $P_{n,1}$ and $P_{n,2}$ for a hollow ball will be as follows:

$$P_{n+1,1} = \int_1^{\rho} \frac{1}{\rho^2} \int_1^{\rho} \rho^2 P_{n,1}\, d\rho d\rho \tag{30}$$

$$P_{n+1,2} = \int_1^{\rho} \frac{1}{\rho^2} \int_1^{\rho} \rho^2 P_{n,2}\, d\rho d\rho \tag{31}$$

$$P_{0,1} = 1; P_{0,2} = 1 - \frac{1}{\rho} \tag{32}$$

For the first quasi-polynomials $P_{1,1}$ and $P_{2.1}$, $P_{1.2}$ and $P_{2.2}$, etc. for a hollow ball you can write:

$$P_{1,1} = \frac{1}{6}\frac{1}{\rho}(\rho-1)^2(\rho+2); P_{2,1} = \frac{1}{120}\frac{1}{\rho}(\rho-1)^4(\rho+4) \tag{33}$$

$$P_{3,1} = \frac{1}{5040}\frac{1}{\rho}(\rho-1)^6(\rho+6); \ldots; \ldots; \tag{34}$$

$$P_{1,2} = \frac{1}{6}\frac{1}{\rho}(\rho-1)^3; P_{2,2} = \frac{1}{120}\frac{1}{\rho}(\rho-1)^5; P_{3,2} = \frac{1}{5040}\frac{1}{\rho}(\rho-1)^7; \ldots; \ldots; \tag{35}$$

Therefore, using the method of mathematical induction, it is possible to write quasi-polynomials for solving the inverse non-stationary problem of heat conduction when setting the boundary condition on the inner surface of a hollow ball in a recurrent form:

$$P_{n,1} = \frac{1}{2n\cdot(2n+1)}(\rho-1)^2 \frac{(\rho+2n)}{(\rho+2(n-1))} P_{n-1,1} \tag{36}$$

$$P_{n,2} = \frac{1}{2n\cdot(2n+1)}(\rho-1)^2 P_{n-1,2} \tag{37}$$

For a given stationary boundary conditions on one surfaces and $\Theta_{n,1}$ and $\Theta_{n,2}$ recurrence relation will be as follows:

$$\Theta_{n,i} = \frac{r_1^2}{a}\frac{\partial \Theta_{n-1,i}}{\partial \tau}, \forall\, i = 1,2 \tag{38}$$

The above relations express the recurrence form of the exact solution of the inverse problem of unsteady heat conduction for bodies of one-dimensional geometry under unsteady boundary conditions specified on one side.

The recurrent form of writing the solution allows us to solve this problem from a single position in a closed form, since expressing solutions in explicit form, as, for example, in [7-19] , is not possible in all cases, as indicated in [2,5,6] .

The questions of the correctness of this inverse heat conduction problem (that is, the existence of a solution, its uniqueness, and its stability) were considered in detail in [5,6]; therefore, this study does not need to be considered again.

The above obtained article solutions nonstationary inverse heat conduction problem for one-dimensional bodies were successfully practical manner employed as part of a conjugate problem when determinancy maximum impact soot layer on the surface of the combustion chamber to the working fluid transient parameters in radiation-convective heat transfer [20,21,22], and also in developing the theory of heat transfer in insulating packaging to stabilize the temperature regimes of storage of perishable products [23,24].

For the heat transfer conditions characteristic of [23,24], calculations were performed for the dependences generated in this article. Under the same temperature boundary condition, the largest deviation will be for a flat body, and the smallest-for a solid ball; for a solid cylinder, there will be an intermediate value.

Both for a hollow cylinder and for a hollow ball, the temperature deviation will be greater than for a solid cylinder and ball, respectively. Comparison of a hollow cylinder with a hollow ball shows that for small values of r_2 / r_1 the deviation for a hollow cylinder will be less than for a hollow ball, but for large values of r_2 / r_1 the deviation for a hollow cylinder will already be greater than for a hollow ball. For these conditions [23,24] above fracture occurs at a value $r_2 / r_1 \approx 3\ ^2/_{15}$. An analysis of the calculations indicates a stronger dependence of the calculated temperature on the parameter r_2 / r_1 for a hollow ball than for a hollow cylinder.

3. Solutions in Recursive Form for Nonstationary Linear Inverse Heat Conduction Problem for Bodies With Dimensional Geometry Boundary Temperature Conditions On Both Surfaces

The temperature fields of hollow cylinders and spheres, plates whose faces are located in different media, are asymmetric, but one-dimensional. An asymmetric temperature field is obtained from measurements of temperatures at the boundaries of the body, which should be previously known functions of time.

The component of the influence of the temperature field of a one-dimensional layer, at the boundaries of which there are non-stationary temperature boundaries, is considered when using a dimensionless coordinate: the first point is taken as the origin, and the second has a unit abscissa (for a flat field); the first point has a unit abscissa, and the second has a point ρ 2 (for spherical and cylindrical fields); can be represented as follows [5]:

$$T(\rho,\mathrm{Fo}) = \sum_{n=0}^{\infty} T_1^{(n)}(\mathrm{Fo})\,P_{n,1} + \sum_{n=0}^{\infty} T_2^{(n)}(\mathrm{Fo})\,P_{n,2} = \sum_{n=0}^{\infty} \Theta_{n,1}\,P_{n,1} + \sum_{n=0}^{\infty} \Theta_{n,2}\,P_{n,2} \quad (39)$$

On both surfaces, there is a boundary condition of the first kind. In this case, temperatures are measured on the boundary surfaces.

Solutions for bodies of simple configuration will differ in the values of the radial quasi-polynomials $P_{n,1}$ and $P_{n,2}$.

In this work, these quasi-polynomials will be solved in the recurrent forms, in contrast to the papers [2,3,4,5,6] and [7-19].

3.1. Flat Plate

The quasi-polynomials $P_{n,1}$ and $P_{n,2}$ for a flat plate will be as follows:

$$P_{n+1,1} = \int_0^\rho \int_0^\rho P_{n,1}\, d\rho d\rho - \rho \int_0^1 \int_0^\rho P_{n,1}\, d\rho d\rho \quad (40)$$

$$P_{n+1,2} = \int_0^\rho \int_0^\rho P_{n,2}\, d\rho d\rho - \rho \int_0^1 \int_0^\rho P_{n,2}\, d\rho d\rho \quad (41)$$

$$P_{0,1} = 1 - \rho; P_{0,2} = \rho \quad (42)$$

For the first quasipolynomials $P_{1,1}$ and $P_{2.1}$, $P_{1.2}$ and $P_{2.2}$, etc. for a flat plate, you can write:

$$P_{1,1} = -\frac{1}{6}\rho^3 + \frac{1}{2}\rho^2 - \frac{1}{3}\rho \quad (43)$$

$$P_{2,1} = -\frac{1}{120}\rho^5 + \frac{1}{24}\rho^4 - \frac{1}{18}\rho^3 + \frac{1}{45}\rho \quad (44)$$

$$P_{3,1} = -\frac{1}{5040}\rho^7 + \frac{1}{720}\rho^6 - \frac{1}{360}\rho^5 + \frac{1}{270}\rho^3 - \frac{2}{945}\rho \quad (45)$$

$$P_{4,1} = -\frac{1}{362880}\rho^9 + \frac{1}{40320}\rho^8 - \frac{1}{15120}\rho^7 + \frac{1}{5400}\rho^5 - \frac{1}{2835}\rho^3 + \frac{1}{4725}\rho; \ldots; \ldots; \quad (46)$$

$$P_{1,2} = \frac{1}{6}\rho^3 - \frac{1}{6}\rho \quad (47)$$

$$P_{2,2} = \frac{1}{120}\rho^5 - \frac{1}{36}\rho^3 + \frac{7}{360}\rho \quad (48)$$

$$P_{3,2} = \frac{1}{5040}\rho^7 - \frac{1}{720}\rho^5 + \frac{7}{2160}\rho^3 - \frac{31}{15120}\rho \quad (49)$$

$$P_{4,2} = \frac{1}{362880}\rho^9 - \frac{1}{30240}\rho^7 + \frac{31}{43200}\rho^5 - \frac{31}{90720}\rho^3 + \frac{127}{604800}\rho; \ldots ; \ldots; \quad (50)$$

Therefore, using the mathematical induction method can be written quasi-polynomials for solving the inverse problem of non-stationary heat conduction by setting the temperature boundary conditions on both boundary surfaces s to the flat plate in recursive form:

$$P_{n,1} = P_{n-1,1} - \frac{1}{(2n+1)!}\rho^{2n+1} + \frac{1}{(2n)!}\rho^{2n} + \sum_{k=0}^{2n-1} \frac{2^{2n+1-k}}{k!(2n-1-k)!}\left(\frac{B_{2n-1-k}}{4} - \frac{1}{(2n-k)}\frac{1}{(2n+1-k)}B_{2n+1-k}\right)\rho^k \quad (51)$$

$$P_{n,2} = P_{n-1,2} - \sum_{k=0}^{2n+1} \frac{(-1)^{4n+2-k}}{k!(2n+1-k)!}\rho^k + \sum_{k=0}^{2n} \frac{(-1)^{4n-k}}{k!(2n-k)!}\rho^k + + \sum_{k=0}^{2n-1} \sum_{l=0}^{k} \frac{(-1)^{4k-l}}{l!(k-l)!} \frac{2^{2n+1-k}}{(2n-1-k)!} \times \left(\frac{B_{2n-1-k}}{4} -- \frac{1}{(2n-k)}\frac{1}{(2n+1-k)}B_{2n+1-k}\right)\rho^l \quad (52)$$

where B_n are the Bernoulli numbers $B_n = \frac{-1}{n+1}\sum_{k=1}^{n} C_{k+1}^{n+1} B_{n-1}, n \in N$: ($C_N^K = \frac{N!}{K!(N-K)!}$ is the binomial coefficient; the number of combinations of N in K) [25] . For example, the first few Bernoulli numbers are equal : $B_0 = 1$; $B_1 = -1 / 2$; $B_2 = 1/6$; $B_3 = 0$; $B_4 = -1 / 30$; $B_5 = 0$; $B_6 = 1/42$; $B_7 = 0$; $B_8 = -1 / 30$; $B_9 = 0$; $B_{10} = 5/66$; $B_{11} = 0$; $B_{12} = -691 / 2730$; $B_{13} = 0$; $B_{14} = 7/6$; $B_{15} = 0$; $B_{16} = -3617 / 510$; $B_{17} = 0$; $B_{18} = 43867/798$; $B_{19} = 0$; $In_{20} = -174611 / 330$ ...

For the last quasi-polynomials $P_{n,2}$, one can rearrange and write it in the following form:

$$P_{n,2} = P_{n-1,2} - \frac{1}{(2n+1)!}(1-\rho)^{2n+1} + \frac{1}{(2n)!}(1-\rho)^{2n} + \sum_{k=0}^{2n-1} \frac{2^{2n+1-k}}{k!(2n-1-k)!} \times \left(\frac{B_{2n-1-k}}{4} - \frac{1}{(2n-k)}\frac{1}{(2n+1-k)}B_{2n+1-k}\right)(1-\rho)^k \quad (53)$$

3.2. Hollow Cylinder

The quasi-polynomials $P_{n,1}$ and $P_{n,2}$ for a hollow cylinder will be as follows:

$$P_{n+1,1} = \int_1^{\rho} \frac{1}{\rho} \int_1^{\rho} \rho P_{n,1}\, d\rho d\rho - \frac{ln\,\rho}{ln\,\rho_2} \int_1^{\rho_2} \frac{1}{\rho} \int_1^{\rho} \rho P_{n,1}\, d\rho d\rho \quad (54)$$

$$P_{n+1,2} = \int_1^{\rho} \frac{1}{\rho} \int_1^{\rho} \rho P_{n,2}\, d\rho d\rho - \frac{ln\,\rho}{ln\,\rho_2} \int_1^{\rho_2} \frac{1}{\rho} \int_1^{\rho} \rho P_{n,2}\, d\rho d\rho \quad (55)$$

$$P_{0,1} = 1 - \frac{ln\,\rho}{ln\,\rho_2}; P_{0,2} = \frac{ln\,\rho}{ln\,\rho_2} \quad (56)$$

For the first quasi-polynomials $P_{1,1}$ and $P_{2.1}$, $P_{1.2}$ and $P_{2.2}$, etc. for a hollow cylinder, you can write:

$$P_{1,1} = \frac{1}{4(ln\,\rho_2)^2} ln\,\rho \left[2(ln\,\rho_2)^2 - \rho_2^2 + 2\,ln\,\rho_2 + 1\right] - \frac{1}{4\,ln\,\rho_2}\left[ln\,\rho_2 + ln\,\rho - \rho^2\,ln\,\rho_2 + \rho^2\,ln\,\rho - \rho^2 + 2\,ln\,\rho_2\,ln\,\rho + 1\right] \quad (57)$$

$$P_{2,1} = \frac{1}{128(ln\,\rho_2)^2}\left[13 ln\,\rho_2 + 8\,ln\,\rho - 8\rho_2^2\,ln\,\rho - 16\rho^2\,ln\,\rho_2 + 3\rho^4\,ln\,\rho_2 ++ 8\rho^2\,ln\,\rho + 8(ln\,\rho_2)^2\,ln\,\rho - 8\rho_2^2 - 8\rho^2 + 6(ln\,\rho_2)^2 + 8\rho_2^2\rho^2 - 8\rho^2(ln\,\rho_2)^2 ++ 2\rho^4(ln\,\rho_2)^2 + 14\,ln\,\rho_2\,ln\,\rho - 8\rho_2^2\rho^2\,ln\,\rho + 8\rho^2\,ln\,\rho_2\,ln\,\rho - 2\rho^4\,ln\,\rho_2\,ln\,\rho ++ 8\right] - \frac{1}{128(ln\,\rho_2)^3} ln\,\rho \left[8\rho_2^4 - 5\rho_2^4\,ln\,\rho_2 - 16\,\rho_2^2\,ln\,\rho_2 - 16\rho_2^2 + 8(ln\,\rho_2)^3 ++ 20(ln\,\rho_2)^2 + 21\,ln\,\rho_2 + 8\right] \quad (58)$$

$$P_{1,2} = \frac{1}{4\,ln\,\rho_2}\left[ln\,\rho + \rho^2\,ln\,\rho - \rho^2 + 1\right] - \frac{1}{4(ln\,\rho_2)^2} ln\,\rho\left[ln\,\rho_2 + \rho_2^2\,ln\,\rho_2 - \rho^2 + 1\right] \quad (59)$$

$$P_{2,2} = \frac{1}{128(ln\,\rho_2)^3} ln\,\rho\left[6\rho_2^4(ln\,\rho_2)^2 - 13\rho_2^4\,ln\,\rho_2 + 8\rho_2^4 + 8\rho_2^2(ln\,\rho_2)^2 - 16\rho_2^2 ++ 6(ln\,\rho_2)^2 + 13\,ln\,\rho_2 + 8\right] - \frac{1}{128(ln\,\rho_2)^2}\left[5\,ln\,\rho_2 + 8\,ln\,\rho + 8\rho_2^2\,ln\,\rho_2 - 8\rho_2^2\,ln\,\rho - 8\rho^2\,ln\,\rho_2 + 3\rho^4\,ln\,\rho_2 ++ 8\rho^2\,ln\,\rho - 8\rho_2^2 - 8\rho^2 + 8\rho_2^2\rho^2 + 6\,ln\,\rho_2\,ln\,\rho - 8\rho_2^2\rho^2\,ln\,\rho_2 - 8\rho_2^2\rho^2\,ln\,\rho ++ 8\rho_2^2\,ln\,\rho_2\,ln\,\rho - 2\rho^4\,ln\,\rho_2\,ln\,\rho + 8\rho_2^2\rho^2\,ln\,\rho_2\,ln\,\rho + 8\right] \quad (60)$$

K vasipolynomials for solving the inverse non-stationary heat conduction problem when setting the boundary temperature conditions on both boundary surfaces for a hollow cylinder in a recurrent form are obtained as follows, based on the solutions obtained for the inverse non-stationary heat conduction problem when setting the boundary condition on the inner surface of the hollow cylinder, i.e. . Formulas (24) and (25).

It is convenient here to introduce local notation, c valid only for this section, in order to avoid further discrepancies in solving the problem:

$$F_{n,1} \stackrel{\text{def}}{=} P_{n,1}\Big|^{(24)} \quad (61)$$

$$F_{n,2} \stackrel{\text{def}}{=} P_{n,2}\Big|^{(25)} \quad (62)$$

In other words, the functions $F_{n,1}$ and $F_{n,2}$ within the framework of this section denote the quasi-polynomials $P_{n,1}$ and $P_{n,2}$ for the inverse non-stationary heat conduction problem with the boundary condition on the inner surface of the hollow cylinder from formulas (24) and (25) respectively .

First, we solve the problem for $P_{n,2}$, since it is simpler than for $P_{n,1}$; solving the first problem will be the basis for solving the second problem.

It's obvious that:

$$P_{0,2} = \frac{F_{0,2}}{ln\,\rho_2} \quad (63)$$

We rewrite the antilaplacians $P_{1,2}$ in the following form:

$$P_{1,2} = \frac{1}{ln\rho_2}\int_1^{\rho}\frac{1}{\rho}\int_1^{\rho}\rho ln\rho\, d\rho d\rho - \frac{ln\rho}{(ln\rho_2)^2}\int_1^{\rho_2}\frac{1}{\rho}\int_1^{\rho}\rho ln\rho\, d\rho d\rho = \frac{1}{ln\rho_2}F_{1,2} - \frac{1}{(ln\rho_2)^2}F_{0,2}F_{1,2}\big|_{\rho=\rho_2} \quad (64)$$

In order to obtain a recursive solution to this problem, we rewrite the last expression as follows:

$$P_{1,2} = \frac{1}{ln\rho_2}F_{1,2} - \frac{1}{ln\rho_2}F_{0,2}\Phi_{1,2}\big|_{\rho=\rho_2} \quad (65)$$

where $\Phi_{1,2} = \frac{1}{\ln\rho_2}F_{1,2}$

The following antilaplacians for the quasi-polynomials $P_{n,2}$ will be as follows:

$$P_{2,2} = \frac{1}{ln\rho_2}F_{2,2} - \frac{1}{ln\rho_2}F_{1,2}\Phi_{1,2}\big|_{\rho=\rho_2} - \frac{1}{ln\rho_2}F_{0,2}\Phi_{2,2}\big|_{\rho=\rho_2}, \quad (66)$$

Where

$$\Phi_{2,2} = \frac{1}{ln\rho_2}F_{2,2} - \frac{1}{ln\rho_2}F_{1,2}\Phi_{1,2}\big|_{\rho=\rho_2}$$

$$P_{3,2} = \frac{1}{ln\rho_2}F_{3,2} - \frac{1}{ln\rho_2}F_{2,2}\Phi_{1,2}\big|_{\rho=\rho_2} - \frac{1}{ln\rho_2}F_{1,2}\Phi_{2,2}\big|_{\rho=\rho_2} - \frac{1}{ln\rho_2}F_{0,2}\Phi_{3,2}\big|_{\rho=\rho_2} \quad (67)$$

where

$$\Phi_{3,2} = \frac{1}{ln\rho_2}F_{3,2} - \frac{1}{ln\rho_2}F_{2,2}\Phi_{1,2}\big|_{\rho=\rho_2} - \frac{1}{ln\rho_2}F_{1,2}\Phi_{2,2}\big|_{\rho=\rho_2}$$

$$P_{4,2} = \frac{1}{ln\rho_2}F_{4,2} - \frac{1}{ln\rho_2}F_{3,2}\Phi_{1,2}\big|_{\rho=\rho_2} - \frac{1}{ln\rho_2}F_{2,2}\Phi_{2,2}\big|_{\rho=\rho_2} - \frac{1}{ln\rho_2}F_{1,2}\Phi_{3,2}\big|_{\rho=\rho_2} - \frac{1}{ln\rho_2}F_{0,2}\Phi_{4,2}\big|_{\rho=\rho_2} \quad (68)$$

where

$$\Phi_{4,2} = \frac{1}{ln\rho_2}F_{4,2} - \frac{1}{ln\rho_2}F_{3,2}\Phi_{1,2}\big|_{\rho=\rho_2} - \frac{1}{ln\rho_2}F_{2,2}\Phi_{2,2}\big|_{\rho=\rho_2} - \frac{1}{ln\rho_2}F_{1,2}\Phi_{3,2}\big|_{\rho=\rho_2}$$

Therefore, antiabrasion the n-th degree for quasi-polynomials PN,2 can be written in the following form:

$$P_{n,2} = \frac{1}{ln\rho_2}F_{n,2} - \sum_{i=0}^{n-1}\frac{1}{ln\rho_2}F_{n-1-i,2}\Phi_{i+1,2}\big|_{\rho=\rho_2} \quad (69)$$

As can be seen from formula (69), for its solution it is used as a “direct” recurrence , i.e. use in the derivation for the current member of a series of previous members of the series, as well as "partial" recurrence, i.e. use in the derivation for the current member of a part of the same member of the series.

Te per should determine the function cp $_{i,2}$. To do this, formalize the form for them. Rewrite the equation (65) for $\Phi_{1,2}$ in the form, characteristic for large values of the parameter i, namely:

$$\Phi_{1,2} = \frac{1}{ln\rho_2}F_{1,2} - \frac{1}{ln\rho_2}F_{0,2}\Phi_{0,2}\big|_{\rho=\rho_2} \quad (70)$$

For the expression $\Phi_{1,2}$ from formula (70) to be identical to its definition from (65) , it is necessary (since $F_{0.2} = \ln\rho$) so that:

$$\Phi_{0,2}\big|_{\rho=\rho_2} = 0. \quad (71)$$

After the last formalization, we can write a closed expression for $\Phi_{i,2}$:

$$\Phi_{i,2} = \frac{1}{ln\,\rho_2} F_{i,2} - \sum_{k=0}^{i-1} \frac{1}{ln\,\rho_2} F_{i-k,2} \Phi_{k,2}\Big|_{\rho=\rho_2} \qquad (72)$$

Thus, expressions (69), (72), (71) give an exact solution quasi-polynomial problems for solving the inverse non-stationary heat conduction problem with specifying temperature boundary conditions on both boundary surfaces for a hollow cylinder in a recurrent form.

As can be seen, the solution for the functions $\Phi_{i,\ 2}$ contains formally terms with $F_{i,2}\Phi_{0,2}\big|_{\rho=\rho_2}$ Obviously, all these terms are absent, for example, in (63) - (69). This is quite natural, since in the accepted representation (72) for $\Phi_{i\ ,\ 2}$ these terms are fictitious and equal (since $\Phi_{0,2}\big|_{\rho=\rho_2} = 0$)) to zero:

In the solution for $P_{n\ ,\ 2}$ (69) there are no terms with $\Phi_{0\ ,\ 2}$, but there are terms with $F_{0\ ,\ 2}$.

In the solution for $\Phi_{i\ ,\ 2}$ (72), there are no terms with $F_{0\ ,\ 2}$, but formally there are terms c that are identically equal to zero. $F_{i,2}\Phi_{0,2}\big|_{\rho=\rho_2} \equiv 0$.

Now you should get a solution for the quasipolynomials $P_{\ n\ ,\ 1}$, using the above solution method and based on existing solutions for $P_{n\ ,\ 2}$, $F_{n\ ,\ 1}$, $F_{n\ ,\ 2}$.

It's obvious that:

$$P_{0,1} = 1 - \frac{ln\,\rho}{ln\,\rho_2} = F_{0,1} - P_{0,2} \qquad (73)$$

C decomposition (54) and (55) for the first terms gives the following expression:

$$P_{1,1} + P_{1,2} = \int_1^{\rho} \frac{1}{\rho} \int_1^{\rho} \rho \left(1 - \frac{ln\,\rho}{ln\,\rho_2}\right) d\rho d\rho - \frac{ln\,\rho}{ln\,\rho_2} \int_1^{\rho_2} \frac{1}{\rho} \int_1^{\rho} \rho \left(1 - \frac{ln\,\rho}{ln\,\rho_2}\right) d\rho d\rho ++$$
$$\int_1^{\rho} \frac{1}{\rho} \int_1^{\rho} \rho \frac{ln\,\rho}{ln\,\rho_2} d\rho d\rho - \frac{ln\,\rho}{ln\,\rho_2} \int_1^{\rho_2} \frac{1}{\rho} \int_1^{\rho} \rho \frac{ln\,\rho}{ln\,\rho_2} d\rho d\rho =$$
$$= \int_1^{\rho} \frac{1}{\rho} \int_1^{\rho} \rho\, d\rho d\rho - \frac{ln\,\rho}{ln\,\rho_2} \int_1^{\rho_2} \frac{1}{\rho} \int_1^{\rho} \rho\, d\rho d\rho = F_{1,1} - \frac{1}{ln\,\rho_2} F_{0,2} F_{1,1}\Big|_{\rho=\rho_2} \qquad (74)$$

In what follows, for $P_{n\ ,\ 1}$, we will do the same in the same way as when solving for the quasi-polynomials $P_{n\ ,\ 2}$, namely:

$$P_{1,1} = F_{1,1} - \frac{1}{ln\,\rho_2} F_{0,2} \Phi_{1,1}\Big|_{\rho=\rho_2} - P_{1,2}, \qquad (75)$$

where $\Phi_{1,1} = F_{1,1}$.

The following antilaplacians for the quasi-polynomials $P_{n\ ,\ 1}$ will be as follows:

$$P_{2,1} = F_{2,1} - \frac{1}{ln\,\rho_2} F_{1,2} \Phi_{1,1}\Big|_{\rho=\rho_2} - \frac{1}{ln\,\rho_2} F_{0,2} \Phi_{2,1}\Big|_{\rho=\rho_2} - P_{2,2} \qquad (76)$$

where $\Phi_{2,1} = \frac{1}{\ln \rho_2} F_{2,1} - \frac{1}{\ln \rho_2} F_{1,2} \Phi_{1,1}\big|_{\rho=\rho_2}$;

$$P_{3,1} = F_{3,1} - \frac{1}{ln\,\rho_2} F_{2,2} \Phi_{1,1}\Big|_{\rho=\rho_2} - \frac{1}{ln\,\rho_2} F_{1,2} \Phi_{2,1}\Big|_{\rho=\rho_2} - \frac{1}{ln\,\rho_2} F_{0,2} \Phi_{3,1}\Big|_{\rho=\rho_2} - P_{3,2} \qquad (77)$$

Where

$$\Phi_{3,1} = F_{3,1} - \frac{1}{ln\,\rho_2} F_{2,2} \Phi_{1,1}\Big|_{\rho=\rho_2} - \frac{1}{ln\,\rho_2} F_{1,2} \Phi_{2,1}\Big|_{\rho=\rho_2};$$

$$P_{4,1} = F_{4,1} - \frac{1}{ln\,\rho_2} F_{3,2} \Phi_{1,1}\Big|_{\rho=\rho_2} - \frac{1}{ln\,\rho_2} F_{2,2} \Phi_{2,1}\Big|_{\rho=\rho_2} - \frac{1}{ln\,\rho_2} F_{1,2} \Phi_{3,1}\Big|_{\rho=\rho_2} -$$
$$\frac{1}{ln\,\rho_2} F_{0,2} \Phi_{4,1}\Big|_{\rho=\rho_2} - P_{4,2} \qquad (78)$$

Where

$$\Phi_{4,1} = F_{4,1} - \frac{1}{ln\,\rho_2}F_{3,2}\Phi_{1,1}\Big|_{\rho=\rho_2} - \frac{1}{ln\,\rho_2}F_{2,2}\Phi_{2,1}\Big|_{\rho=\rho_2} - \frac{1}{ln\,\rho_2}F_{1,2}\Phi_{3,1}\Big|_{\rho=\rho_2}$$

Consequently, antilaplasiany n - o rd degree for quasi-polynomials $P_{n,1}$ can be written as follows:

$$P_{n,1} = F_{n,1} - \sum_{i=0}^{n-1}\frac{1}{ln\,\rho_2}F_{n-1-i,2}\Phi_{i+1,1}\Big|_{\rho=\rho_2} - P_{n,2} \quad (79)$$

Te per should determine the function cp $_{i,1}$. To do this, formalize the form for them. Rewrite the equation (75) fo r$\Phi_{1,1}$ a, which is characteristic for the used ólshih values of the parameter i , namely:

$$\Phi_{1,1} = \frac{1}{ln\,\rho_2}F_{1,1} - \frac{1}{ln\,\rho_2}F_{0,2}\Phi_{0,1}\Big|_{\rho=\rho_2} \quad (80)$$

For the expression $\Phi_{i,1}$ from formula (80) to be identical to its definition from (75), it is necessary (since $F_{0,2}$ = ln ρ) so that:

$$\Phi_{0,1}\Big|_{\rho=\rho_2} = 0 \quad (81)$$

After the last formalization, we can write a closed expression for $\Phi_{i,1}$:

$$\Phi_{i,1} = F_{i,1} - \sum_{k=0}^{i-1}\frac{1}{ln\,\rho_2}F_{i-k,2}\Phi_{k,1}\Big|_{\rho=\rho_2} \quad (82)$$

Thus, expressions (79), (82), (81) give an exact solution quasi-polynomial problems for solving the inverse non-stationary heat conduction problem with specifying temperature boundary conditions on both boundary surfaces for a hollow cylinder in a recurrent form.

As can be seen, the solution for the functions $\Phi_{i,1}$ contains formally terms with $F_{i,2}\Phi_{0,1}\big|_{\rho=\rho_2}$, Obviously, all these terms are absent, for example, in (73) - (82). This is quite natural, since in the accepted representation (82) for Φ i , 1 these terms are fictitious and equal (since $\Phi_{0,1}\big|_{\rho=\rho_2} = 0$) to zero: $F_{i,2}\Phi_{0,1}\big|_{\rho=\rho_2} \equiv 0$.

In the solution for $P_{n,1}$ (79) there are no terms with $\Phi_{i,1}$, but there are terms with $F_{0,2}$.

In the solution for $\Phi_{i,1}$ (82) there are no terms with $F_{0,2}$, but formally there are terms c that are identically equal to zero.

In principle, the problem of exact solution of the quasi-polynomials $P_{n,2}$ - (69), (72), (71) - and Pn, 1 - (79), (82), (81) - for the inverse non-stationary problem of heat conduction when setting the temperature boundary conditions on both boundary surfaces for a hollow cylinder in recurrent form can be completed . O dnako , these solutions can be written in a united form, which should be rewritten in the form of rows corresponding to $\Phi_{i,1}$ and $\Phi_{i,2}$ of the formulas (82) and (72), respectively:

$$\Phi_{i+1,1} = F_{i+1,1} - \sum_{k=0}^{i}\frac{1}{ln\,\rho_2}F_{i+1-k,2}\Phi_{k,1}\Big|_{\rho=\rho_2} \quad (83)$$

$$\Phi_{i+1,2} = \frac{1}{ln\,\rho_2}F_{i+1,2} - \sum_{k=0}^{i}\frac{1}{ln\,\rho_2}F_{i+1-k,2}\Phi_{k,2}\Big|_{\rho=\rho_2} \quad (84)$$

In Ob e dinonnoy form accurate solutions of this problem (for $R_{n,1}$ - (79) and $P_{n,2}$ - (69)) will be as follows:

$$P_{n,1} = F_{n,1} - \sum_{i=0}^{n-1}\frac{1}{ln\,\rho_2}F_{n-1-i,2}\left[F_{i+1,1} - \sum_{k=0}^{i}\frac{1}{ln\,\rho_2}F_{i+1-k,2}\Phi_{k,1}\Big|_{\rho=\rho_2}\right]\Bigg|_{\rho=\rho_2} - P_{n,2} \quad (85)$$

$$-\sum_{i=0}^{n-1}\frac{1}{ln\rho_2}F_{n-1-i,2}\left[\frac{1}{ln\rho_2}F_{i+1,2}-\sum_{k=0}^{i}\frac{1}{ln\rho_2}F_{i+1-k,2}\Phi_{k,2}\Big|_{\rho=\rho_2}\right]\Big|_{\rho=\rho_2} \quad (86)$$

3.3. Hollow Ball

The quasipolynomials $P_{n,1}$ and $P_{n,2}$ for a hollow ball will be as follows:

$$P_{n+1,1}=\int_1^\rho\frac{1}{\rho^2}\int_1^\rho\rho^2P_{n,1}\,d\rho d\rho-\frac{\rho_2}{\rho}\frac{(\rho-1)}{(\rho_2-1)}\int_1^{\rho_2}\frac{1}{\rho^2}\int_1^\rho\rho^2P_{n,1}\,d\rho d\rho \quad (87)$$

$$P_{n+1,2}=\int_1^\rho\frac{1}{\rho^2}\int_1^\rho\rho^2P_{n,2}\,d\rho d\rho-\frac{\rho_2}{\rho}\frac{(\rho-1)}{(\rho_2-1)}\int_1^{\rho_2}\frac{1}{\rho^2}\int_1^\rho\rho^2P_{n,2}\,d\rho d\rho \quad (88)$$

$$P_{0,1}=\frac{1}{(\rho_2-1)}\left(\frac{\rho_2}{\rho}-1\right);P_{0,2}=\frac{\rho_2}{(\rho_2-1)}\left(1-\frac{1}{\rho}\right) \quad (89)$$

For the first quasi-polynomials $P_{1,1}$ and $P_{2.1}$, $P_{1.2}$ and $P_{2.2}$, etc. for a hollow ball you can write:

$$P_{1,1}=\frac{4}{3}\frac{1}{\rho}(\rho_2-1)^2\left(-\frac{1}{4}\frac{(\rho-1)}{(\rho_2-1)}+\frac{3}{8}\frac{(\rho-1)^2}{(\rho_2-1)^2}-\frac{1}{8}\frac{(\rho-1)^3}{(\rho_2-1)^3}\right) \quad (90)$$

$$P_{2,1}=\frac{4}{15}\frac{1}{\rho}(\rho_2-1)^4\left(\frac{1}{12}\frac{(\rho-1)}{(\rho_2-1)}-\frac{5}{24}\frac{(\rho-1)^3}{(\rho_2-1)^3}+\frac{5}{32}\frac{(\rho-1)^4}{(\rho_2-1)^4}-\frac{1}{32}\frac{(\rho-1)^5}{(\rho_2-1)^5}\right) \quad (91)$$

$$P_{3,1}=\frac{8}{315}\frac{1}{\rho}(\rho_2-1)^6\left(-\frac{1}{12}\frac{(\rho-1)}{(\rho_2-1)}+\frac{7}{48}\frac{(\rho-1)^3}{(\rho_2-1)^3}-\frac{7}{64}\frac{(\rho-1)^5}{(\rho_2-1)^5}+\frac{7}{128}\frac{(\rho-1)^6}{(\rho_2-1)^6}-\frac{1}{128}\frac{(\rho-1)^7}{(\rho_2-1)^7}\right) \quad (92)$$

$$P_{4,1}=\frac{4}{2835}\frac{1}{\rho}(\rho_2-1)^8\left(\frac{3}{20}\frac{(\rho-1)}{(\rho_2-1)}-\frac{1}{4}\frac{(\rho-1)^3}{(\rho_2-1)^3}+\frac{21}{160}\frac{(\rho-1)^5}{(\rho_2-1)^5}-\frac{3}{64}\frac{(\rho-1)^7}{(\rho_2-1)^7}+\frac{9}{512}\frac{(\rho-1)^8}{(\rho_2-1)^8}-\frac{1}{512}\frac{(\rho-1)^9}{(\rho_2-1)^9}\right);\ldots; \quad (93)$$

$$P_{1,2}=\frac{4}{3}\frac{\rho_2}{\rho}(\rho_2-1)^2\left(-\frac{1}{4}\frac{(\rho_2-\rho)}{(\rho_2-1)}+\frac{3}{8}\frac{(\rho_2-\rho)^2}{(\rho_2-1)^2}-\frac{1}{8}\frac{(\rho_2-\rho)^3}{(\rho_2-1)^3}\right) \quad (94)$$

$$P_{2,2}=\frac{4}{15}\frac{\rho_2}{\rho}(\rho_2-1)^4\left(\frac{1}{12}\frac{(\rho_2-\rho)}{(\rho_2-1)}-\frac{5}{24}\frac{(\rho_2-\rho)^3}{(\rho_2-1)^3}+\frac{5}{32}\frac{(\rho_2-\rho)^4}{(\rho_2-1)^4}-\frac{1}{32}\frac{(\rho_2-\rho)^5}{(\rho_2-1)^5}\right) \quad (95)$$

$$P_{3,2}=\frac{8}{315}\frac{\rho_2}{\rho}(\rho_2-1)^6\left(-\frac{1}{12}\frac{(\rho_2-\rho)}{(\rho_2-1)}+\frac{7}{48}\frac{(\rho_2-\rho)^3}{(\rho_2-1)^3}-\frac{7}{64}\frac{(\rho_2-\rho)^5}{(\rho_2-1)^5}+\frac{7}{128}\frac{(\rho_2-\rho)^6}{(\rho_2-1)^6}-\frac{1}{128}\frac{(\rho_2-\rho)^7}{(\rho_2-1)^7}\right) \quad (96)$$

$$P_{4,2}=\frac{4}{2835}\frac{\rho_2}{\rho}(\rho_2-1)^8\left(\frac{3}{20}\frac{(\rho_2-\rho)}{(\rho_2-1)}-\frac{1}{4}\frac{(\rho_2-\rho)^3}{(\rho_2-1)^3}+\frac{21}{160}\frac{(\rho_2-\rho)^5}{(\rho_2-1)^5}-\frac{3}{64}\frac{(\rho_2-\rho)^7}{(\rho_2-1)^7}+\frac{9}{512}\frac{(\rho_2-\rho)^8}{(\rho_2-1)^8}-\frac{1}{512}\frac{(\rho_2-\rho)^9}{(\rho_2-1)^9}\right);\ldots; \quad (97)$$

Therefore, using the method of mathematical induction, it is possible to write quasi-polynomials for solving the inverse non-stationary problem of heat conduction when specifying the temperature boundary conditions on both boundary surfaces for a hollow ball in a recurrent form:

$$P_{n,1}=P_{n-1,1}-\frac{1}{(2n+1)!}\frac{1}{(\rho_2-1)}\frac{1}{\rho}(\rho-1)^{2n+1}+\frac{1}{(2n)!}\frac{1}{\rho}(\rho-1)^{2n}+\frac{1}{\rho}\sum_{k=0}^{2n-1}\frac{2^{2n+1-k}}{k!(2n-1-k)!}(\rho_2-1)^{2n-2-k}(\rho-1)^k\times\left(\frac{B_{2n-1-k}}{4}-(\rho_2-1)^2\frac{1}{(2n-k)}\frac{1}{(2n+1-k)}B_{2n+1-k}\right) \quad (98)$$

$$P_{n,2}=P_{n-1,2}-\frac{1}{(2n+1)!}\frac{1}{(\rho_2-1)}\frac{\rho_2}{\rho}(\rho_2-\rho)^{2n+1}+\frac{1}{(2n)!}\frac{\rho_2}{\rho}(\rho_2-\rho)^{2n}$$

$$+\frac{\rho_2}{\rho}\sum_{k=0}^{2n-1}\frac{2^{2n+1-k}}{k!(2n-1-k)!}(\rho_2-1)^{2n-2-k}(\rho_2-\rho)^k\times\left(\frac{B_{2n-1-k}}{4}-(\rho_2-1)^2\frac{1}{(2n-k)}\frac{1}{(2n+1-k)}B_{2n+1-k}\right) \quad (99)$$

For given nonstationary temperature boundary conditions on both surfaces $\Theta_{n,1}$ and $\Theta_{n,2}$, the recurrence relations are equal to:

$$\Theta_{n,i}=\frac{r_1^2}{a}\frac{\partial\Theta_{n-1,i}}{\partial\tau}, \forall\, i=1,2 \quad (100)$$

4. Conclusions

The relevance of the problem of solving the inverse linear non-stationary heat conduction problem of a one-dimensional geometric shape, obtained in this work in a closed recursive form, consists in the fact that it is possible to restore the boundary conditions from the measurements of the heat flux sensor with a sufficient degree of accuracy.

In this paper, we obtained exact analytical solutions for the unsteady linear inverse heat conduction problem for bodies of one-dimensional geometry with boundary conditions on one surface, as well as on two surfaces for a plane body and hollow cylinders and spheres, obtained in a recurrent form.

The recurrence form of the solution obtained in the paper for the solution of the unsteady linear inverse heat conduction problem for bodies of one-dimensional geometry with boundary conditions on one surface, as well as on two surfaces for a plane body, a hollow cylinder, and a hollow sphere, is a closed-form solution from a single position, which not always possible explicitly.

From a practical point of view, the obtained solutions can be used in the calculation of non-stationary temperature fields and heat flux densities for various materials used in aviation and rocket and space technology, based on the measured non-stationary boundary conditions on one of the sides, as well as on two surfaces for flat body, hollow cylinder and hollow sphere.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this article.

Funding

The work is supported by the Moscow Aviation Institute.

References

[1] Stefan, J. Ueber die Theorie der Eisbildung, insbesondere über die Eisbildung im Polarmeere. *Annalen der physik,* 1891, 278(2), 269-286.

[2] Burggraf, O.R. An exact solution of the inverse problem in the theory of thermal conductivity and its applications. *Transactions of the American Society of Mechanical Engineers. Series C: Heat Transfer,* 1964, 3, 94-106.

[3] Langford, D. New analytical Solutions of the One – Dimensional Heat Equation for Temperature are Heat Flow Rate Both Prescribed at the Same Fixed Boundary

(with applications to the phase change problem). *Q. App. Math,* 1976, 24(4), 315-322.

[4] Imber, M.; Khan, D. Calculation of unsteady temperature distribution on Based on indications tepmopar disposed inside the body. *Missile technology and astronautics,* 1972, 2, 83-90.

[5] Tyomkin, A.G. Inverse heat conduction problems. In book: Heat Conduction - Basic Research, 1973; pp. 464, DOI: 10.5772/26575.

[6] Beck, D.W.; Blackwell, B.; Saint Clair Ch ml. Incorrect inverse heat conduction problems. *M.: Mir,* 1989; pp. 312.

[7] Kavtaradze, R.Z.; Lapushkin, N.A.; Lobanov, I.E. Investigation of the heat-insulating effect of a carbon layer on the surfaces of a diesel engine using the inverse and conjugate methods of thermal conductivity. *Izv, universities. Engineering,* 1997, 4(6), 66-71.

[8] Kavtaradze, R.Z.; Lapushkin, N.A.; Lobanov, I.E. Calculation and experimental study of unsteady heat transfer in the combustion chamber of a high-speed diesel engine taking into account the heat-insulating action of the carbon layer. Improving the power, economic and environmental performance of ICE. Materials of the VI international scientific and practical seminar. Vladimir, 1997; pp. 111-112.

[9] Kavtaradze, R.Z.; Lapushkin, N.A.; Lobanov, I.E. Investigation of the heat-insulating effect of a carbon layer using inverse and conjugate methods of thermal conductivity. Engine-97. Materials of the international scientific and technical conference. 1997; pp. 25.

[10] Lobanov, I.E. An analytical solution to the nonlinear inverse heat conduction problem for a body with a low coefficient of thermal conductivity of one-dimensional geometry. Problems of gas dynamics and heat and mass transfer in power plants: Program of the XII School-Seminar for Young Scientists and Specialists under the supervision of Academician A.I. Leontiev. M.: MPEI, 1999; pp. 10.

[11] Lobanov, I.E. Exact analytical solutions of the nonlinear non-stationary inverse heat conduction problem for a body with a low coefficient of heat conductivity. Proceedings of universities. *Aircraft technology,* 2010, 3, 72-74.

[12] Lobanov, I.E. Inverse one-dimensional nonlinear heat conduction problem : exact analytical solutions. *Thermophysics and Heat Engineering,* 2012, 1(1), 3-12.

[13] Lobanov, I.E. Exact analytical solutions of the nonlinear non-stationary inverse problem of heat conduction for bodies with a low coefficient of heat conductivity of one-dimensional geometry. *Almanac of modern science and education,* 2010, 8(39), 56-64.

[14] Lobanov, I.E. Nonlinear non-stationary inverse heat conduction problem for bodies of one-dimensional geometry with a low coefficient of thermal conductivity: exact analytical solutions. *Thermal processes in engineering,* 2012, 4(6), 274-283.

[15] Lobanov, I.E. Exact analytical solutions of the nonlinear non-stationary inverse heat conduction problem for bodies with a low coefficient of thermal conductivity of one-dimensional geometry. Transactions of the XIV Minsk International

Forum on Heat and Mass Transfer. Minsk, 2012. Section No. 7. General issues of heat and mass transfer and heat conduction. Report, No. 1-19, S. 1-11.

[16] Lobanov, I.E. Exact analytical solutions of nonlinear nonstationary inverse heat conduction problem for bodies with a low coefficient of thermal conductivity of one-dimensional geometrY. Theses of reports and messages XIV Minsk International Forum on Heat o- and Mass Transfer. *Minsk,* 2012, 1(2), 729-732.

[17] Lobanov, I.E. Theoretical and experimental determination of the unsteady temperature state of a soot layer in the combustion chambers of heat engines. *SCI-ARTICLE.RU,* 2016, 40, 194-206.

[18] Lobanov, I.E. To issue determinancy influence copper pl g APIS in surface temperature sensor design on calc g t heat state soot layer. *SCI-ARTICLE.RU,* 2017, 43, 142-148.

[19] Lobanov, I.E.; Paramonov, N.V. Measurement and modeling of thermal loads in the chambers of internal combustion engines. M.: And the building of the Moscow Aviation Institute, 2012; pp. 160.

[20] Lobanov, I.E.; Dotsenko, A.I. The influence of the carbon layer on the surfaces of the combustion chambers on the parameters of the working fluid. *Mechanization of Construction A,* 2009, 5, 23-26.

[21] Lobanov, I.E. Theoretical determination of the maximum effect of a carbon layer on the surface of a combustion chamber on unsteady parameters of a working fluid during radiation-convective heat transfer. *M Oskov Scientific Review,* 2013, 9, 11-15.

[22] Lobanov, I.E. Calcd g spot-experimental method of indirect measurement of the thickness of the coke layer on the surfaces of heat engines combustion chambers. *SCI-ARTICLE.RU,* 2016, 38, 96-100.

[23] Belozerov, G.A.; Lobanov, I.E. The use of heat-insulating packaging to stabilize the temperature conditions of storage of perishable products. *Actual problems of modern science,* 2012, 2, 193-200.

[24] Lobanov, I.E. The theory of heat transfer of heat-insulating packaging to stabilize the temperature conditions of storage of perishable products. *Studies of Technical Sciences,* 2011, 1(1), 3-10.

[25] Beitman, G.; Erdeyi, A. Higher transcendental functions. Hypergeometric function. Legendre functions. M.: Publishing house “Science”. The main edition of the physical and mathematical literature, 1965; pp. 296.