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Applications of Kutta-Joukowski Transformation to the Flow Past a Flat Plate

Sa Pai Myint[1*]

[1] Mathematics Department, University of Yangon, Yangon, Myanmar

Email Address

teachersapai@uy.edu.mm (Sa Pai Myint)
*Correspondence: teachersapai@uy.edu.mm

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Abstract:

In this paper, conformal mapping is mainly applied to the flow past a flat plate for the airfoil. Kutta-Joukowski transformation is expressed. The applications of Joukowski transformation are discussed. Kutta-Joukowski lift theorem is also proved. The force acting on a solid body is discussed. Lift formula for the flat plate is obtained. Forces on the cylinders are derived. Conformal mapping is explained and is applied to the two-dimensional flows. Joukowski transformation which is also conformal is explained and is applied to the flows. Kutta-Joukowski lift theorem is also proved. Flows past arbitrary airfoils are considered by using Joukowski transformation.

Keywords:

Conformal Transformation, Airfoil , Kutta-Joukowski Transformation

1. Introduction

Conformal transformation is an important technique used in complex analysis and has many applications in different physical situations. If the physical problem can be represented by complex functions but geometric structure becomes inconvenient then by an appropriate mapping it can be transformed to a problem with much more convenient geometry. In the following section, this article gives a brief introduction to conformal mappings and some of its applications in physical problems are presented. Many complicated flow boundaries may be transformed into regular flow boundaries, such as the ones already studied, by the technique of conformal transformations. The theory of conformal mapping tells us that it is possible to map any simply connected profile onto a circular cylinder. In the transformation, the flow past the profile maps into the flow past a circular cylinder and far field flow is preserved. Therefore the inverse transformation describes the exact solution of flow past the original profile. One such transformation is the Joukowski transformation

With the Joukowski transformation of the obstacle to a cylinder that preserves that the far field, the lift over the obstacle is the same as the lift over the cylinder at the same angle of incidence. The Kutta-Joukowski transformation lift theorem is a fundamental theorem in aerodynamics and for the circulation of lift of an airfoil and two-dimensional bodies including circular cylinders. This theorem is applied to two-dimensional flow around a fixed airfoil. Joukowski transformation which is also conformal is applied to the flow past a flat plate.

2. The Fluid Flow Outside a Circular Cylinder

2.1. Circle Theorem

Let f(z) be the complex potential for a flow having no rigid boundaries and such that there is no singularities of flow z within the circle $|z| = a$. Then on introducing the solid circular cylinder $|Z| = a$ into the flow, the new complex potential is given by:

$$W = f(z) + \overline{f}\left(\frac{a^2}{\overline{z}}\right), \ |z| \geq a \tag{1}$$

2.2. Conformal Transformation

The mapping t = f (z) is said to be *conformal* at a point z_0 if the function f(z) is analytic at z_0 and $f'(z_0) \neq 0$.

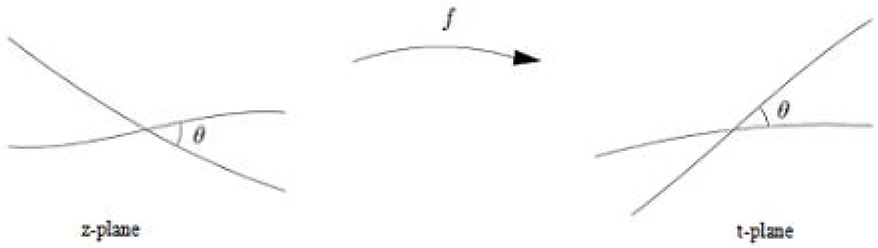

Figure 1. *The nature of conformal.*

2.3. The Joukowski Transformation

The transformation $Z = z + \frac{a^2}{z}$ is one of the most important transformations used for the two-dimensional motion. Firstly we shall apply it to the region outside the circle $|z| = b$, where b > a, then to the region outside the circle $|z| = a$ and finally to a general circle.

2.4. Uniform Flow Past a Cylinder

The complex potential for a stream U flowing in the negative direction of x-axis is easily seen to be Uz. On inserting a circular cylinder $|z| = a$ into the stream, the complex potential for the resulting motion is found from the circle theorem to be

$$w = Uz + \frac{Ua^2}{z}. \tag{2}$$

$$\phi + i\psi = U(re^{i\theta} + \frac{a^2}{re^{i\theta}});$$

$$\phi + i\psi = U(r\cos\theta + ir\sin\theta + \frac{a^2}{r}\cos\theta - i\frac{a^2}{r}\sin\theta);$$

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$$\phi + i\psi = U\cos\theta(r + \frac{a^2}{r}) + iU\sin\theta(r - \frac{a^2}{r});$$

The stream function and velocity potential are

$$\psi = U(r - \frac{a^2}{r})\sin\theta;$$

$$\phi = U(r + \frac{a^2}{r})\cos\theta;$$

the velocity

$$-u + iv = \frac{dw}{dz} = U(1 - \frac{a^2}{z^2});$$

therefore the stagnation points are $z = a$ and $z = -a$. At a point on the cylinder, $z = ae^{i\alpha\theta}$

$$-q_r + iq_\theta = \frac{dw}{dr} e^{i\theta} = U\,(1 - \frac{a^2 e^{-2i\theta}}{a^2})$$

$$= U(e^{i\theta} - e^{-i\theta}) = 2iU\sin\theta.$$

Here, $q_r = 0$ *and* $q_\theta = 2iU\sin\theta$.

That is , the velocity is tangential and of magnitude $2U\sin\theta$.

If the stream makes an angle α with Ox, its complex potential is $U z e^{-i\alpha}$, therefore for a cylinder in an oblique stream

$$w = U\,(z e^{-i\alpha} + \frac{a^2 e^{i\alpha}}{z}). \tag{3}$$

By a change of origin, if the centre of cylinder is at the point z0 ,

$$w = U\,(z e^{-i\alpha} + \frac{a^2 e^{i\alpha}}{z - z_0}). \tag{4}$$

If there is a circulation κ about the cylinder in the positive sense, then we must add a term $(\frac{i\kappa}{2\pi})$ log (z) to each of the equations (1) and (2) and $(\frac{i\kappa}{2\pi})$ log ($z - z_0$) to (4).

2.5. Flow with Circulation Past a Flat Plate

The complex potential for a fixed circulation cylinder of radius a in a stream whose undisturbed speed U makes an angle α with the x-axis and about which there is a circulation κ, is

$$w = U(ze^{-i\alpha} + \frac{a^2 e^{i\alpha}}{z}) + \frac{i\kappa}{2\pi}\log z. \tag{5}$$

If the Jouskowski transformation $Z = z + \frac{a^2}{z}$ is applied to the whole area outside the circle in the z-plane, it transforms into the whole of the Z-plane with a rigid barrier between the points (± 2a, 0). Then the problem becomes that of a flat plate of width 4a, about which there is a circulation, in a stream U inclined at α to the plate.

Jouskowski transformation can be Solve for z in terms of Z as follow,

$$Z = z + \frac{a^2}{z}$$

$$zZ = z^2 + a^2$$

$$-a^2 = z^2 - zZ$$

$$-4a^2 = 4z(z - Z)$$

$$Z^2 - 4a^2 = 4z^2 - 4zZ + Z^2 = (Z - 2z)^2$$

$$(Z - 2z) = \pm\sqrt{Z^2 - 4a^2}$$

$$Z = 2z \pm \sqrt{Z^2 - 4a^2}$$

$$z = \frac{1}{2}\left(Z \mp \sqrt{Z^2 - 4a^2}\right)$$

we have

$$z = \frac{1}{2}\left(Z + \sqrt{Z^2 - 4a^2}\right)$$

and

$$z = \frac{1}{2}\left(Z - \sqrt{Z^2 - 4a^2}\right)$$

taking the positive sign in the first expression, since z is outside the circle $|z| = a$. Hence the complex potential (1) becomes:

$$\begin{aligned} w &= \frac{1}{2}U\left\{\left(Z + \sqrt{Z^2 - 4a^2}\right)e^{-i\alpha} + \left(Z - \sqrt{Z^2 - 4a^2}\right)e^{i\alpha}\right\} \\ &+ \frac{i\kappa}{2\pi}\log\left(Z + \sqrt{Z^2 - 4a^2}\right) \\ &= U\left\{Z\cos\alpha - i\left(\sqrt{Z^2 - 4a^2}\right)\sin\alpha\right\} \\ &+ \frac{i\kappa}{2\pi}\log\left(Z + \sqrt{Z^2 - 4a^2}\right), \end{aligned}$$

neglecting a constant. The circulation about the plate is given by the decrease in the velocity potential ϕ on describing a circuit round it; this is the same as the decrease in ϕ on describing the corresponding circuit about the cylinder, i.e., there is a circulation κ about the plate.

The velocity at any point can be written:

$$\begin{aligned} -U + iV &= \frac{dw}{dZ} \\ &= \frac{dw}{dz}\cdot\frac{dz}{dZ} \\ &= \frac{dw}{dz}\Big/(1 - \frac{a^2}{z^2}). \end{aligned} \tag{6}$$

The denominator vanishes when $z = \pm\ a$ i.e., $z = \pm\ 2a$, therefore the velocity is infinite at both edges unless $\frac{dw}{dz}$ has a factor $(z + a)$ or $(z - a)$, when it will be finite at the corresponding edge. Investigating this, we find from (4):

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$$\frac{dw}{dz} = U(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}) + \frac{i\kappa}{2\pi z}.$$

If this is zero when z = a,

$$\frac{dw}{dz} = U(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}) + \frac{i\kappa}{2\pi z} = 0$$

$$U(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}) + \frac{i\kappa}{2\pi z} = 0$$

$$U(\cos\alpha - i\sin\alpha - \frac{a^2}{z^2}\cos\alpha - i\frac{a^2}{z^2}\sin\alpha) + \frac{i\kappa}{2\pi z} = 0$$

z = a

$$U(-2i\sin\alpha) + \frac{i\kappa}{2\pi z} = 0.$$

then

$$\kappa = 4\pi a\, U \sin\alpha.$$

Hence if this condition holds, the velocity at the edge Z = 2a will be finite.

3. Lift on the Airfoil

3.1. Kutta-Joukowski Lift Theorem

The theory of conformal mapping gives that it is possible to map any simply connected profile onto a circular cylinder. In the transformation, the flow past the profile maps into the flow past a circular cylinder derived earlier. The far field flow is preserved. The inverse transformation will therefore describe the exact solution of the flow past the original profile.

One such transformation is the Joukowski transformation

$$x = X\left(1 + \frac{a^2}{X^2 + Y^2}\right), \quad y = Y\left(1 - \frac{a^2}{X^2 + Y^2}\right), \tag{7}$$

where (x, y) represents the physical plane of the profile and (X, Y) the plane of the cylinder. Notice that the transformation preserves the far field (i.e., $x \to X$ and $y \to Y$ far from the origin).

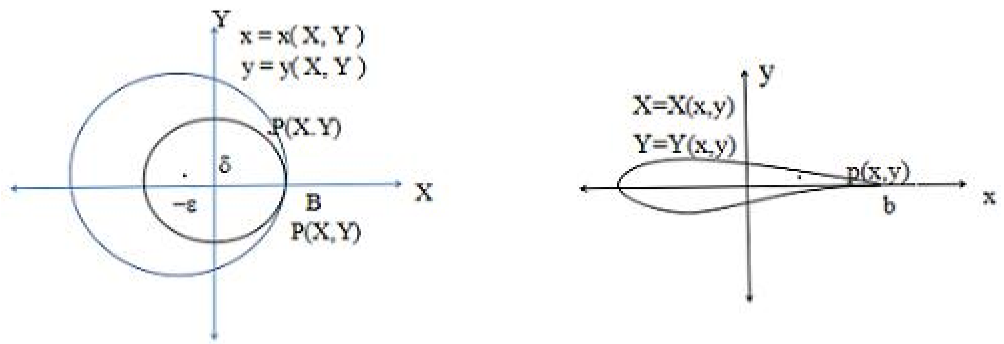

Figure 2. *Cylinder plane (X,Y) and physical plane (x,y).*

A cylinder of radius r_0, where $r_0^2 = (a+\varepsilon)^2 + \delta^2$ centered at $X = -\varepsilon$, $Y = \delta$ and passing through the point B located at $X = a$, $Y = 0$ maps onto a family of

Joukowski airfoils depending on ε and δ. The profile has a cusp at the trailing edge, point b, image of point B. In Figure 2, p is the image of P through the transformation.

The polar coordinate representation of the cylinder is

$$X = r(\theta)\cos\theta,\ Y = r(\theta)\sin\theta, \tag{8}$$

where:

$$r(\theta) = -(\varepsilon\cos\theta - \delta\sin\theta) + \sqrt{a^2 + 2\varepsilon a + (\varepsilon\cos\theta - \delta\sin\theta)^2}.$$

Through the mapping, the parametric representation of the Joukowski profile is obtained.

3.2. Applications of Joukowski Transformation

3.2.1. The Flat Plate at Incidence

The Joukowski's transformation is given by

$$x(\theta) = \left(r(\theta) + \frac{a^2}{r(\theta)}\right)\cos\theta,$$

$$y(\theta) = \left(r(\theta) - \frac{a^2}{r(\theta)}\right)\sin\theta,$$

Where:

$$r(\theta) = -(\varepsilon\cos\theta - \delta\sin\theta) + \sqrt{a^2 + 2\varepsilon a + (\varepsilon\cos\theta - \delta\sin\theta)^2}.$$

For this case, Joukowski transformation has $\varepsilon = \delta = 0$. Thus r (θ) = a. The flat plate is mapped into a circle with its center at the origin in the (X,Y) -plane, Figure 3. The mapping can be interpreted as a conformal transformation from the cylinder plane to the physical plane such that $x = \Phi / U$, $y = \Psi / U$, where Φ and Ψ are the velocity potential and steam function of the flow past a circular cylinder without circulation. For this flow, we have

$$\Phi = (Ur + \frac{\mu}{r})\cos\theta \quad \text{and} \quad \Psi = (Ur - \frac{\mu}{r})\sin\theta.$$

Therefore,

$$x = \frac{\Phi}{U} = \left(r + \frac{a^2}{r}\right)\cos\theta = X\left(1 + \frac{a^2}{X^2 + Y^2}\right) \tag{9}$$

and

$$y = \frac{\Psi}{U} = \left(r - \frac{a^2}{r}\right)\sin\theta = Y\left(1 - \frac{a^2}{X^2 + Y^2}\right). \tag{10}$$

In this transformation, the cylinder of radius a centered at the origin, which corresponds to the stagnation streamline Ψ = 0, is mapped onto the x-axis (y = 0), in the interval $[-2a, 2a]$.

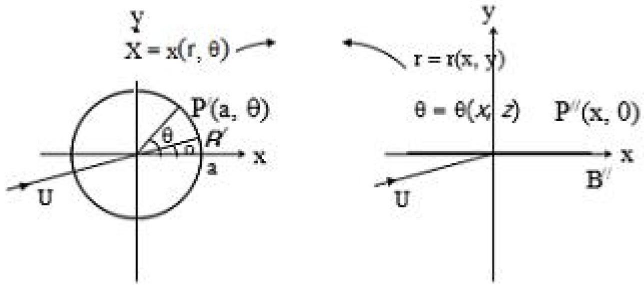

Figure 3. *Flat plate transformation.*

This case corresponds to $b = a = \frac{c}{4}$ in $[-2a,\ 2a]$, and the results derived previously can be used.

In the mapping, the flat plate parametric representation is given by:

$$x = 2a\cos\theta = \frac{c}{2}\cos\theta,\quad y = 0 \tag{11}$$

and the velocity components in the physical plane are

$$\left.\begin{aligned} \frac{\partial\Phi}{\partial x} &= 2U\frac{\left(\sin(\theta-\alpha)+\dfrac{\Gamma}{4\pi a}\right)}{\sin\theta}, \\ \frac{\partial\Phi}{\partial y} &= 0. \end{aligned}\right\} \tag{12}$$

Forcing the circulation to be zero yields an anti-symmetric flow that does not satisfy the Kutta-Joukowski condition (Figure 4). We obtained the moment coefficient as $C_{m,o} = \frac{\pi}{2}\left(1-\left(\frac{e}{c}\right)^2\right)\sin\alpha\,\cos\alpha$ and for the flat $\frac{e}{c} = 0$ plate the lift is zero, but the moment coefficient about the mid-plate reduces to

$$C_{m,o} = \frac{\pi}{2}\sin\alpha\cos\alpha. \tag{13}$$

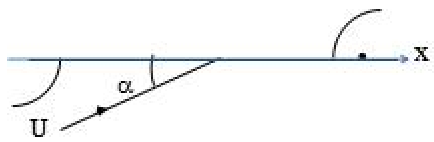

Figure 4. *Flat plate at 25° incidence (Γ = 0).*

Making use of the mapping, the velocity on the flat plate now is

$$\left.\begin{aligned} \frac{\partial\Phi}{\partial x} &= 2U\left(\cos\alpha \mp \sin\alpha\frac{\frac{2x}{c}}{\sqrt{1-\left(\frac{2x}{c}\right)^2}}\right), \\ \frac{\partial\Phi}{\partial y} &= 0. \end{aligned}\right\} \tag{14}$$

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where the minus (plus) sign corresponds to the upper (lower) surface. The stagnation points are at $x = \frac{c}{2}\cos$, $y = 0$.

When the Kutta-Joukowski condition is applied, the velocity on the flat plate becomes

$$\frac{\partial \Phi}{\partial x} = U\frac{\cos\left(\frac{\theta}{2}-\alpha\right)}{\cos\frac{\theta}{2}}, \quad \frac{\partial \Phi}{\partial y} = 0 \cdot \tag{15}$$

The streamline in this case leaves the trailing edge 'smoothly', (Figure 4).

We note that the velocity is infinite at $\theta = \pi$ (leading edge) in all cases except for $\alpha = 0$. The stagnation point corresponds to $\theta = \pi + 2\alpha$, i.e., $x = \frac{c}{2}\cos 2\alpha$. It is also worthy to note that the exact solution is consistent with the thin airfoil theory result, since the flat plate is the ultimate thin airfoil, having zero thickness and zero camber. Another assumption that will be made is that of small α. If we map the flat plate to the segment [0, c] with $x' = \frac{c}{2}\left(1+\frac{x}{2a}\right)$, the horizontal velocity component becomes

$$\frac{\partial \Phi}{\partial x} = U\left(\cos\alpha \pm \sin\alpha\sqrt{\frac{c-x'}{x'}}\right) \simeq U\left(1 \pm \alpha\sqrt{\frac{c-x'}{x'}}\right). \tag{16}$$

The plus (minus) sign corresponds to the upper (lower) surface. This last result is the thin airfoil result with a perturbation u above and below the plate that is symmetric about the x'-axis.

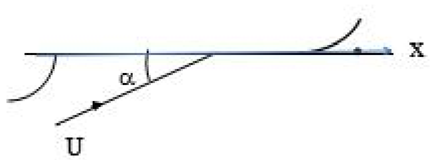

Figure 5. *Flat plate at 25° incidence (κ= 4π U a sin α).*

The pressure coefficient in the uniform flow $q_\infty = (U, 0)$ is given by $C_p = 1-\left(\frac{q}{q_\infty}\right)^2$, where $q = \frac{\partial \phi}{\partial x}$ and q_∞

Therefore,

$$C_p = 1-\left(\frac{1}{U}\frac{\partial \phi}{\partial x}\right)^2. \tag{17}$$

The result for $-C_p$ is displayed in Figure 6. It is not symmetric with respect to the Ox'-axis. If we anticipate again on thin airfoil theory, the linearized C_p will be symmetric, i.e. $C_p^- = -C_p^+$.

The lift coefficient is obtained from the Kutta-Joukowski lift theorem

$$C_l = \frac{\rho U \kappa}{\frac{1}{2}\rho U^2 4a} = 2\pi\sin\alpha \cdot \tag{18}$$

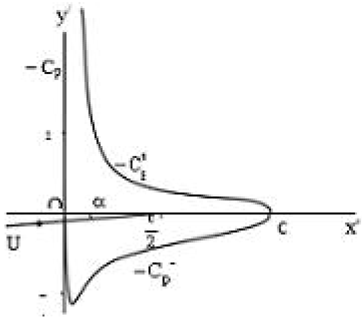

Figure 6. *Pressure coefficients along the plate at $\alpha = 10°$.*

It is interesting, however, to consider the integration of the pressure along the flat plate directly. The result is expected to be normal to the plate, i.e., in the y - direction. Indeed, integrating in θ, the elementary force due to pressure around the plate yields the pressure contribution $\underline{F}'_p = (0, F'_p)$, where

$$F'_p = 4\pi \rho U^2 a \sin\alpha \cos\alpha . \tag{19}$$

On the other hand, the lift per unit span, as given by the Kutta - Joukowski theorem is

$$L' = 4\pi\rho U^2 a \sin\alpha \tag{20}$$

acting in a direction perpendicular to the incoming velocity vector as $\underline{L}' = (-L'\sin\alpha, L'\cos\alpha)$. The difference between these results is due to a horizontal suction force $\underline{F}'_s = (F'_s, 0)$ acting at the leading edge, in the direction of the negative x-axis, that is, due to the leading edge singularity. This missing component, parallel to the camber line and of magnitude

$$F'_s = -4\pi\rho U^2 a \sin^2\alpha \tag{21}$$

points to the fact that integration of pressure does not capture the localized force that exists at a sharp leading edge.

The moment with respect to the leading edge is the sum of elementary moments

$$d_{M,O} = -\frac{1}{2}\rho U^2 C_p(\theta)(x(\theta) + 2a)\, dx(\theta) . \tag{22}$$

Here, O stands for the leading edge. The moment is given by

$$M_{,O} = 2\rho U^2 a^2 \int_0^{2\pi} \left(1 - \left(\frac{\cos\left(\frac{\theta}{2} - \alpha\right)}{\cos\frac{\theta}{2}} \right)^2 \right) (1 + \cos\theta)\sin\theta \, d\theta . \tag{23}$$

Some simplifications occur when the half-angle $\frac{\theta}{2}$ is used. The result is

$$M_{,O} = -2\pi\rho U^2 a^2 \sin 2\alpha, \tag{24}$$

and the moment coefficient reads

$$C_{m,o} = -\frac{\pi}{4}\sin 2\alpha \tag{25}$$

The moment of the aerodynamic forces can be calculated at an arbitrary point D along the x'- axis as $C_{m,D} = C_{m,o} + \frac{x'_D}{c} C_l$. The center of pressure satisfies $C_{m,c.p.} = 0$, i.e.

$$\frac{x'_{c.p.}}{c} = -\frac{C_{m,o}}{C_l} = \frac{1}{4}\cos\alpha \simeq \frac{1}{4}. \tag{26}$$

For small values of α, the center of pressure for the flat plate is located at the quarter-chord.

Taking the derivative with respect to α of the same general formula for the moment at point D and now setting the result to zero, gives the aerodynamics center

$$\frac{\partial C_{m,D}}{\partial \alpha} = -\frac{\pi}{2}\cos 2\alpha + \frac{x'_D}{c} 2\pi \cos\alpha = 0. \tag{27}$$

For small values of α, one obtains the small perturbation result

$$\frac{x'_{a.c.}}{c} = \frac{1}{4} \tag{28}$$

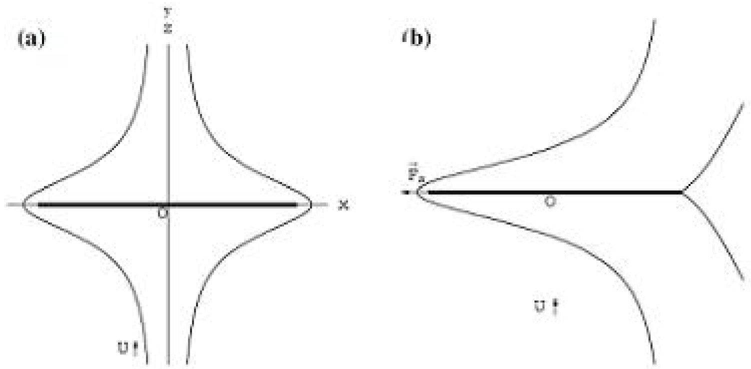

***Figure 7 .** Flat plate perpendicular to incoming flow (a) with zero circulation (b) with trailing edge that satisfies the K-J condition and suction force.*

4. Conclusions

The limiting case of the flow coming perpendicular to the plate ($\alpha = 90°$) is considered, with the plate having zero circulation or satisfying the Kutta-Joukowski condition. The two results are shown in Figure 7.

In Figure 7(a), the flow is perfectly symmetric about the mid-plate and the net resulting force and moments are zero. Again, this is an idealized flow representation and viscosity will radically change the picture by having separation at the sharp edges of the plate. The same can be said of Figure 7 (b) in terms of realism of the picture, but it is intriguing to observe that there is circulation, hence lift, and the force is not perpendicular to the flat plate, but aligned with the plate and to the left.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this article.

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