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New Proofs of Two Problems in Mathematical Analysis

Asiyeh Erfanmanesh[1*]

[1] Institute For Higher Education, ACECR, Khouzestan, Iran

Email Address

erfanmaneshasy@gmail.com (Asiyeh Erfanmanesh)
*Correspondence: erfanmaneshasy@gmail.com

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Abstract:

First, a simple proof of the known result that the set of irrational numbers cannot be written as a countable union of closed sets in R, is given. Furthermore, we present an interesting example to show that product of two closed subsets of a topological groups need not be always closed.

Keywords:

Nowhere Dense Set, Topological Group, Additive Group

1. A Problem in Classical Analysis

As we know mathematical analysis has key role in solving problems in physics, engineering and so on. It is addressed to undergraduate and beginning graduate students of mathematics, physics, and engineering who want to learn how functional analysis elegantly solves mathematical problems that are related to our real world and that have played an important role in the history of mathematics.

In the present section we give a new and simple proof of the following well-known result of classical analysis.(See, for example [2]).Throughout the literature, $\mathfrak{R}$, Q^c, Q, $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of real, irrational, rational, integer and natural numbers, respectively.

Problem 1. The set of irrational numbers can not be written as a countable union of closed sets in $\mathfrak{R}$.

Proof. Let by contrary,

$$Q^c = \bigcup_{n=1}^{\infty} F_n$$

where any F_n, $n \in \mathbb{N}$, is a closed set in $\mathfrak{R}$. We set $Q = \{r_1, r_2, ...\}$. Then, evidently $\mathfrak{R} = \bigcup_{n=1}^{\infty} (F_n \cup \{r_n\})$. It is obvious that $F_n \cup \{r_n\}$ is closed, for any $n \in \mathbb{N}$. We claim that $(F_n \cup \{r_n\})^o = \emptyset$, for any $n \in \mathbb{N}$. Equivalently, we show that $((F_n \cup \{r_n\})^o)^c = \mathfrak{R}$, for any $n \in \mathbb{N}$. To see this, we show that $\overline{(F_n \cup \{r_n\})^c} = \mathfrak{R}$, for any $n \in \mathbb{N}$. Note that one can also write Equality as $Q = \bigcap_{n=1}^{\infty} F_n^c$. Then by taking closure of the both sides of the last equality we get

$$R = \bigcap_{n=1}^{\infty} \overline{F_n^c}$$

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It follows that

$$\overline{F_n^c} = R\,,$$

for any $n \in \mathbb{N}$. Thus we have

$$\overline{(F_n \cup \{r_n\})^c} = \overline{F_n^c} = R$$

for any $n \in N$. This establishes our claim. Therefore, $\mathfrak{R}$ is as a countable union of its nowhere dense subsets which is a contradiction.

2. A Problem in Topological Groups

Topological groups, along with continuous group actions, are used to study continuous symmetries, which have many applications, for example, in physics. In physics, a symmetry of a physical system is a physical or mathematical feature of the system (observed or intrinsic) that is preserved or remains unchanged under some transformation. [3,4,5]. A family of particular transformations may be continuous (such as rotation of a circle) or discrete (e.g., reflection of a bilaterally symmetric figure, or rotation of a regular polygon). Continuous and discrete transformations give rise to corresponding types of symmetries. Continuous symmetries can be described by Lie groups while discrete symmetries are described by finite groups (see Symmetry group). These two concepts, Lie and finite groups, are the foundation for the fundamental theories of modern physics. Symmetries are frequently amenable to mathematical formulations such as group representations and can, in addition, be exploited to simplify many problems. Arguably the most important example of a symmetry in physics is that the speed of light has the same value in all frames of reference, which is known in mathematical terms as the Poincaré group, the symmetry group of special relativity. Another important example is the invariance of the form of physical laws under arbitrary differentiable coordinate transformations, which is an important idea in general relativity.

Now, we present a problem in topological groups.

In [1], it has been mentioned that the product of two closed subsets of a topological groups in not necessary closed. In the following, we give a simple example for this problem.

Problem 2. Let G be a topological group, A and B be closed subsets of G. Then AB need not be closed.

Proof. Consider the additive group $\mathfrak{R}$ and closed subsets $\mathbb{Z}$ and $\alpha\mathbb{Z}$, where α is an arbitrary irrational number. Notice that $\mathbb{Z} + \alpha\mathbb{Z} = \{m + \alpha n{:}m,n \in \mathbb{Z}\}$. We claim that the set $\mathbb{Z} + \alpha\mathbb{Z}$ is not a closed subset of $\mathfrak{R}$.For see this, firstly we define the following sequence of elements of $\mathbb{Z} + \alpha\mathbb{Z}$:

$$x_n = \frac{p[\sqrt{n(n+1)}]}{q\sqrt{n(n+1)}}$$

hewre p and $0 \neq q$ are integers such that $(p,q) = 1$ and [x] denotes the integer part of x. It is easy to show that $\sqrt{n(n+1)}$ is irrational, for any $n \in \mathbb{N}$. Since $\sqrt{n(n+1)} \sim [\sqrt{n(n+1)}]$ as $n \to \infty$, so $x_n \to \frac{p}{q}$ as $n \to \infty$. This proves our claim. Here, in addition, we show that $\mathbb{Z} + \alpha\mathbb{Z}$ is dense in $\mathfrak{R}$. Suppose that $y \in \mathfrak{R}$. Consider the following sequence of elements of $\mathbb{Z} + \alpha\mathbb{Z}$:

$$y_n = \frac{[y\sqrt{n(n+1)}]}{\sqrt{n(n+1)}}$$

As before, $y_n \to y$ as $n \to \infty$. Therefore, $\mathbb{Z} + \alpha\mathbb{Z}$ is dense in $\Re$, as desired.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this article.

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